Calculus of Fractions for ∞-Categories

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This article is an effort to generalize the corresponding parts in Kashiwara-Schapira's book *Categories and Sheaves*, in which they consider the case of 1-categories. Besides that, the main reference is Lurie's *Higher Topos Theory*, abbreviated as HTT. We will always refer to the version on his personal website.

Contents

1 Calculus of Fractions

1.1 Localization by Right Fractions

To take localization of an ∞-category is to make some of its morphisms invertible, in a free or universal way.

Definition 1.1.1. *Let S be a class of morphisms in an ∞-category $\mathcal{C}$. A functor $f : \mathcal{C} \to \mathcal{C}'$ is S-invariant if it sends morphisms in S to equivalences. We will write $\mathrm{Fun}_S(\mathcal{C}, \mathcal{C}')$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ comprised of S-invariant functors. A localization of $\mathcal{C}$ at S is an S-invariant functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that it induces equivalence of categories for any $\mathcal{C}'$:*

$$q^* : \mathrm{Fun}(S^{-1}\mathcal{C}, \mathcal{D}') \simeq \mathrm{Fun}_S(\mathcal{C}, \mathcal{C}')$$

The localization of a small ∞-category $\mathcal{C}$ at any class S is always exist since it is nothing but a homotopy pushout in $\mathcal{C}\mathrm{at}_\infty$:

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{C} \\ \big\downarrow & & \big\downarrow \\ |S| & \longrightarrow & S^{-1}\mathcal{C} \end{array}$$

Here we regard S as a subcategory of $\mathcal{C}$ and $S \to |S|$ denotes the geometric realization. This description leads to a useful property that, if we regard S as morphisms of $h\,\mathcal{C}$, we have canonical equivalence $S^{-1}(h\,\mathcal{C}) \simeq h(S^{-1}\mathcal{C})$ since taking homotopy categories commutes with small colimits. We will not develop any general theory about localization but focus on a specific case that is more tractable but still useful.

Given any S-invariant functor $q : \mathcal{C} \to \mathcal{C}'$, we are able to construct a functor $q_X : S_{X/} \to \mathcal{C}_{/q(X)}$ for any $X \in \mathcal{C}$ that sends $s : X \to X' \in S_{X/}$ to $q(s)^{-1} : q(X') \to q(X) \in \mathcal{C}_{/q(X)}$. These functors can be exploited to identify a special kind of localization functors.

Theorem 1.1.2. *Assume that an S-invariant functor $q : \mathcal{C} \to \mathcal{C}'$ between small ∞-categories satisfies the following properties:*

1. *The functor q is essentially surjective;*

2. *For any object $X \in \mathcal{C}$, the functor q_X is cofinal.*

Then q is a localization functor $\mathcal{C} \to S^{-1}\mathcal{C} \simeq \mathcal{C}'$.

Proof. The theorem follows from two claims: (a) given any S-invariant functor $f : \mathcal{C} \to \mathcal{D}$, the left Kan extension f' of f along q exists and it holds that $f' \circ q \simeq f$; (b) any functor $f' : \mathcal{C}' \to \mathcal{D}$ is the left Kan extension of its composition $f' \circ q$. Actually (b) can be deduced from (a) using the universal property together with the essential surjectivity of q. Also by the essential surjectivity, to prove (a), we only need to show that, given any S-invariant functor $f : \mathcal{C} \to \mathcal{D}$, the colimit of $\mathcal{C}_{/q(X)} \to \mathcal{C} \to \mathcal{D}$ exists and is naturally equivalent to $f(X)$. Since q_X is cofinal, after composing with q_X, we can consider the corresponding diagram $p : S_{X/} \to \mathcal{D}$. By the definition of $S_{X/}$ and the S-invariance of f, we see that p sends all morphisms to equivalences and the contractibility of $S_{X/}$ (since it has initial object id_X) guarantees that this diagram is equivalent to a constant digram with value $f(X)$. □

Given any functor $f : \mathcal{C} \to \mathcal{C}'$ and $X \in \mathcal{C}'$, the right fibration $\mathcal{C}_{/X} \to \mathcal{C}$ is classified by the presheaf:

$$\mathrm{Map}_{\mathcal{C}'}(f(-), X) \simeq \mathrm{colim}_{f(X') \to X \in \mathcal{C}_{/X}} \mathrm{Map}_{\mathcal{C}}(-, X')$$

It follows that Theorem 1.1.2 implies the mapping spaces of $S^{-1}\mathcal{C}$ can be represented by the colimits:

$$\mathrm{Map}_{S^{-1}\mathcal{C}}(A, X) \simeq \mathrm{colim}_{X \to X' \in S_{X/}} \mathrm{Map}_{\mathcal{C}}(A, X')$$

More precisely, fixing $A \in \mathcal{C}$, the co-presheaf $\mathrm{Map}_{\mathcal{C}}(A, -)$ over $S_{X/}$ corresponds to a left fibration:

$$\mathrm{Span}_S(A, X) \to S_{X/}$$

The ∞-category $\mathrm{Span}_S(A, X)$ is the subcategory of $\mathrm{Fun}(\Lambda^2_0, \mathcal{C})$ comprised of objects $A \to X' \leftarrow X$ such that $X' \to X \in S$, which we call right fractions, and morphisms of the following form:

$$\begin{array}{ccccc} A & \longrightarrow & X' & \xleftarrow{s'} & X \\ {\scriptstyle id}\downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\ A & \longrightarrow & X'' & \xleftarrow{s''} & X \end{array}$$

Therefore the previous colimit formula has an equivalent formulation:

$$\mathrm{Map}_{S^{-1}\mathcal{C}}(A, X) \simeq |\,\mathrm{Span}_S(A, X)|$$

In particular, after taking π_0, we deduce that any morphism in $S^{-1}\mathcal{C}$ can be factorized as $q(s)^{-1} \circ q(r)$ such that $s \in S$ and the corresponding right fractions of different factorizations are connected by zig-zags of morphisms in $\mathrm{Span}_S(A, X)$.

If we impose furthermore a filteredness condition, we will obtain a generalization of classical calculus of fractions for 1-categories.

Definition 1.1.3. *Let S be a class of morphisms of $\mathcal{C}$ that contains all equivalences and is closed under composition. If the following holds:*

1. *For any object $X \in \mathcal{C}$, the ∞-category $S_{X/}$ is filtered.*

2. *There exists a functor q satisfies Theorem 1.1.2;*

We say that S admits calculus of right fractions.

Remark 1.1.4. *This definition captures the most important property of the classical notion and allows to generalize the good results to ∞-categories, in a relatively easy way. It seems that our notion, even in the case of 1-categories, is a bit more general than the classical one. However, we will show that they are equivalent in (cf. Theorem 1.4.6) in the end of this chapter.*

The last result in this section is a criterion to decide whether S admits calculus of right fractions, which will be used later.

Theorem 1.1.5. *Let S be a class of morphisms of $\mathcal{C}$ that contains all equivalences, is closed under composition and for any $X \in \mathcal{C}$, the ∞-category $S_{X/}$ is filtered. Then S admits calculus of right fractions if and only if the following presheaf is S-invariant for any $X \in \mathcal{C}$:*

$$\varinjlim_{X \to X' \in S_{X/}} \mathrm{Map}_{\mathcal{C}}(-, X')$$

Or equivalently, it is an S-local object of $\mathrm{ind}\text{-}\mathcal{C}$.

Proof. We have to show that there exists a functor satisfying Theorem 1.1.2. The diagram $f_X : S_{X/} \to \mathcal{C}$, seen as an ind-object of $\mathcal{C}$ which we will denote as $LX \in \mathrm{ind}\text{-}\mathcal{C}$, is S-local by assumption. Since id_X is the initial object of $S_{X/}$, we have a morphism $p_X : X \to LX$. Actually p_X is the S-localization of X in $\mathrm{ind}\text{-}\mathcal{D}$. This is because p_X is S-equivalence since p_X is the filtered colimit of morphisms in S.

By the universal property of S-localization, we obtain a functor $L : \mathcal{C} \to \mathrm{ind}\text{-}\mathcal{C}$ and a natural transformation $j \to L$ (j denotes the Yoneda embedding). We will write $\mathcal{C}'_{/LX}$ and $\mathcal{C}_{/LX}$ respectively for the comma fiber of j and L over LX. Also by the universal property, we have equivalence $\mathcal{C}'_{/LX} \simeq \mathcal{C}_{/LX}$. By (.....!!!!!!!!), the canonical functor $S_{X/} \to \mathcal{C}'_{/LX} \simeq \mathcal{C}_{/LX}$ is cofinal, and the functor is equivalent to the q_X mentioned in Theorem 1.1.2. The functor we want is the restriction of L to its essential image. $\square$

1.2 Compatibility with Finite Colimits and Zero Objects

Provided that S admits calculus of fractions, the localization procedure is well compatible with finite colimits.

Theorem 1.2.1. *Let S be a class of morphisms that admits calculus of right fractions. The localization functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ is right exact.*

Proof. The functor $q_X : S_{X/} \to \mathcal{C}_{/q(X)}$ is cofinal and $S_{X/}$ is filtered, and hence $\mathcal{C}_{/q(X)}$ is filtered. $\square$

Theorem 1.2.2. *Let S be a class of morphisms that admits calculus of right fractions. The following holds:*

1. *If $\mathcal{C}$ admits initial objects, so is $S^{-1}\mathcal{C}$;*
2. *If $\mathcal{C}$ admits finite coproducts, so is $S^{-1}\mathcal{C}$;*
3. *If $\mathcal{C}$ admits pushouts, so is $S^{-1}\mathcal{C}$;*
4. *If $\mathcal{C}$ admits finite colimits, so is $S^{-1}\mathcal{C}$;*

Moreover the localization functor preserves any finite colimits that exist.

Proof. The last claim, (1) and (2) follows immediately from the right exactness and essential surjectivity of $q : \mathcal{C} \to S^{-1}\mathcal{C}$. The claim (4) is a corollary of (1), (2) and (3). We are left to prove (3). The remark above implies that any morphism $q(A) \to q(B)$ in $S^{-1}\mathcal{C}$ can be factorized as $q(s)^{-1} \circ q(r)$ for some $s \in S$. Therefore given any diagram as the first row in the following diagram, we can have an equivalence between diagrams:

$$\begin{array}{ccccc} q(X) & \xleftarrow{f} & q(A) & \xrightarrow{f'} & q(X') \\ {\scriptstyle q(s)}\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle q(s')} \\ q(Y) & \xleftarrow[q(r)]{} & q(A) & \xrightarrow[q(r')]{} & q(Y') \end{array}$$

The pushout of the second row exists since it is from $\mathcal{C}$, and hence is the first row. $\square$

One can use a similar argument to show the following.

Theorem 1.2.3. *Let $S \subseteq \mathcal{C}$ be a class of morphisms that admits calculus of right fractions. If $\mathcal{C}$ admits finite colimits, then for any right exact S-invariant functor $f : \mathcal{C} \to \mathcal{C}'$, the canonical functor given by universal property $\tilde{f} : S^{-1}\mathcal{C} \to \mathcal{C}'$ is also right exact.*

Remark 1.2.4. *All of the results in this section have κ-version for any uncountable regular cardinal κ if we assume that $S_X/$ is κ-filtered whatever we take $X \in \mathcal{C}$.*

In general, one should not hope that calculus of right fractions have compatibility with limits, with one exception.

Theorem 1.2.5. *Let $S \subseteq \mathcal{C}$ be a class of morphisms that admits calculus of right fractions. If $\mathcal{C}$ admits zero objects, $S^{-1}\,\mathcal{C}$ admits zero objects and the localization functor preserves them.*

Proof. ... $\square$

1.3 Localization of Subcategories

Let S be a class of morphisms of $\mathcal{C}$ that admits calculus of right fractions, $\mathcal{C}_0 \subseteq \mathcal{C}$ a full subcategory and S_0 denotes the class $S \cap \mathcal{C}_0$.

Theorem 1.3.1. *Suppose that for any object $X \in \mathcal{C}_0$ and morphism $X \to X' \in S$, there exists $X' \to X''$ such that $X'' \in \mathcal{C}_0$ and the composition $X \to X' \to X''$ is in S. Then S_0 admits calculus of right fractions, and the canonical functor $S_0^{-1}\,\mathcal{C}_0 \to S^{-1}\,\mathcal{C}$ is fully faithful.*

Proof. The assumption implies that for any $X \in \mathcal{C}_0$, $(S_0)_{X/}$ is filtered and the inclusion $(S_0)_{X/} \to S_{X/}$ is cofinal (cf. !!!!!!!!!!). Let $q : \mathcal{C} \to S^{-1}\,\mathcal{C}$ be the localization functor. It follows that $q|_{\mathcal{C}_0}$ seen as functor to its essential image satisfies Theorem 1.1.2. □

1.4 Ore Condition

In this section, we give a generalization of Ore condition to ∞-categories. The material here is merely for the purpose of integrity and will not be used later. We begin with some notations. Let D_n denote the simplicial set $\Delta^0 \star \partial\Delta^n \star \Delta^0$, D_n^+ its simplicial subset $\emptyset \star \partial\Delta^n \star \Delta^0$, D_n^- its simplicial subset $\Delta^0 \star \partial\Delta^n \star \emptyset$ and E_n their intersection $D_n^+ \cap D_n^- \simeq \emptyset \star \partial\Delta^n \star \emptyset$. Also we denote two special 1-simplexes $\Delta^0 \star \{0\} \star \emptyset$ and $\emptyset \star \{n\} \star \Delta^0$ of D_n as f^+ and f^-. The cone points of D_n^+ and D_n^- will be denoted simply as $+$ and $-$.

Definition 1.4.1. *Let S be a class of morphisms of a small ∞-category. We say S satisfies right Ore condition if for any diagram ($n \geq 1$):*

$$\begin{array}{ccc} D_n^+ & \xrightarrow{p} & \mathcal{C} \\ \downarrow & \nearrow_{\tilde{p}} & \\ D_n & & \end{array}$$

such that $p|_{f^+}$ is in S, there exists $\tilde{p}$ such that $\tilde{p}|_{f^-}$ is also in S. If moreover S contains all equivalences and is closed under composition, we call it a right multiplicative system of $\mathcal{C}$.

Remark 1.4.2. *This notion is equivalent to the classical one if $\mathcal{C}$ is 1-category. In particular, if S is a right multiplicative system of $\mathcal{C}$, it is also a right multiplicative system of $h\,\mathcal{C}$.*

Lemma 1.4.3. *Given a right multiplicative system S, $S_{X/}$ is filtered for any object $X \in \mathcal{C}$.*

Proof. By the extension property in the definition. $\square$

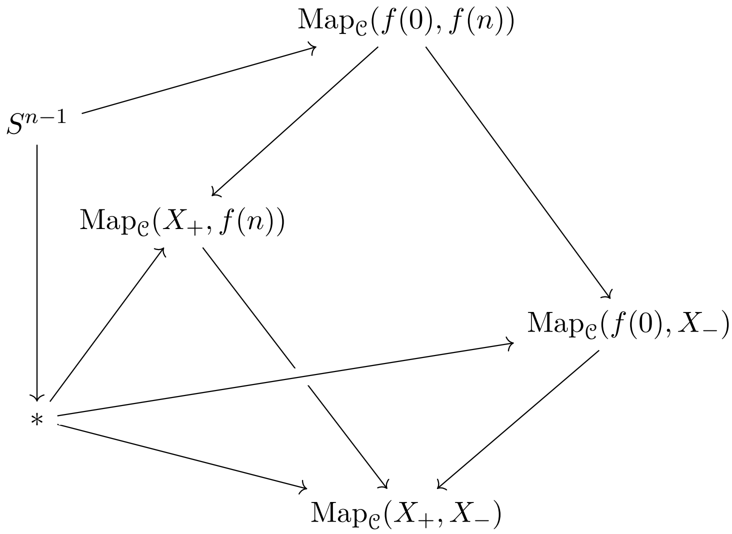
We need an alternative description of the extension property about D_n. Any diagram $f : E_n \to \mathcal{C}$ provides a map $S^{n-1} \to \mathrm{Map}_{\mathcal{C}}(f(0), f(n))$. An extension from E_n to D_n^+ is equivalent to take a morphism $X_+ \to f(0)$ and a null-homotopy:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(f(0), f(n)) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X_+, f(n)) \end{array}$$

Similarly an extension from E_n to D_n^- is equivalent to take a morphism $f(n) \to X_-$ and a null-homotopy:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(f(0), f(n)) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(f(0), X_-) \end{array}$$

A further extension to D_n is equivalent to take a large polyhedral diagram with the two top squares the given null-homotopies and the right square given by the functoriality of mapping spaces:

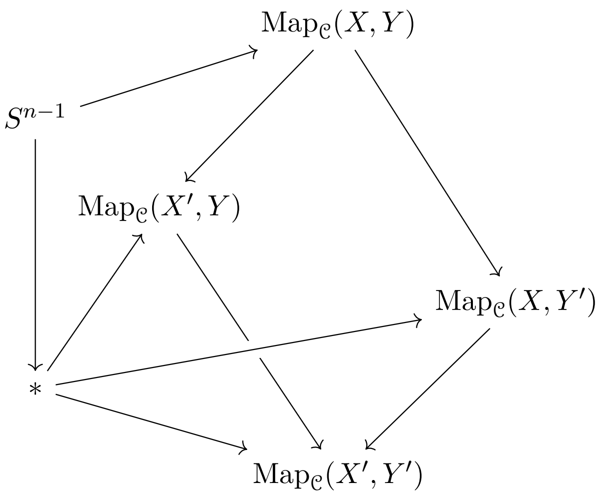


The previous discussion leads to the following lemma.

Lemma 1.4.4. *Right Ore condition for S is equivalent to that, given any map $S^{n-1} \to \mathrm{Map}_{\mathcal{C}}(X, Y)$, morphism $X' \to X \in S$ and null-homotopy square:*

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, Y) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X', Y) \end{array}$$

There exists $Y \to Y' \in S$ and an extension of the given square to the polyhedron:



If we take the fibers of the squares in the previous lemma, we can obtain an equivalent formulation in a somewhat easier form.

Corollary 1.4.5. *Right Ore condition for S is equivalent to that, given any morphism $s : X \to X' \in S_{X/}$ and map $S^{n-1} \to \mathrm{Map}_{\mathcal{C}_{X/}}(s, f)$ (we write $f : X \to Y$), there exist $s' : Y \to Y' \in S$ such that the composition $S^{n-1} \to \mathrm{Map}_{\mathcal{C}_{X/}}(s, f) \to \mathrm{Map}_{\mathcal{C}_{X/}}(s, s'f)$ is null-homotopic.*

Moreover, if $S_{Y/}$ is filtered for all $Y \in \mathcal{C}$, the previous condition is equivalent to that $\varinjlim_{s':Y \to Y' \in S_{Y/}} \mathrm{Map}_{\mathcal{C}_{X/}}(s, s'f) \simeq *$ holds.

Theorem 1.4.6. *Being right multiplicative system is equivalent to admitting calculus of right fractions.*

Proof. According to Theorem 1.1.5 and Whitehead theorem, admitting calculus of right fractions is equivalent to that, given any $A \to B \in S$, the following extension problems always have solutions in $\mathcal{S}$:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \varinjlim_{X \to X' \in S_{X/}} \mathrm{Map}_{\mathcal{C}}(B, X') \\ \downarrow & \nearrow & \downarrow \\ * & \longrightarrow & \varinjlim_{X \to X' \in S_{X/}} \mathrm{Map}_{\mathcal{C}}(A, X') \end{array}$$

Taking the fibers, and using the fact that filtered colimits commute with finite limits, we observe that the fibers are always contractible is equivalent to the last condition in Corollary 1.4.5. $\square$

1.5 Recognition of Localization Functor

We will use the notation E_n, f_+, f_-, D_n, D_n^+ and D_n^- introduced in the beginning of the previous section. The main result in this part will be a criterion to decide when a functor q is a localization functor with respect to a calculus of right fractions.

Given a functor $q : \mathcal{C} \to \mathcal{C}'$, and a class S of morphisms of $\mathcal{C}$ that contains all equivalences, is closed under composition and for any $X \in \mathcal{C}$, the ∞-category $S_{X/}$ is filtered, we have the following criterion.

Theorem 1.5.1. *Assume that q is a categorical fibration. Then S admits right calculus of fractions and q is a localization functor at S if and only if the following conditions hold:*

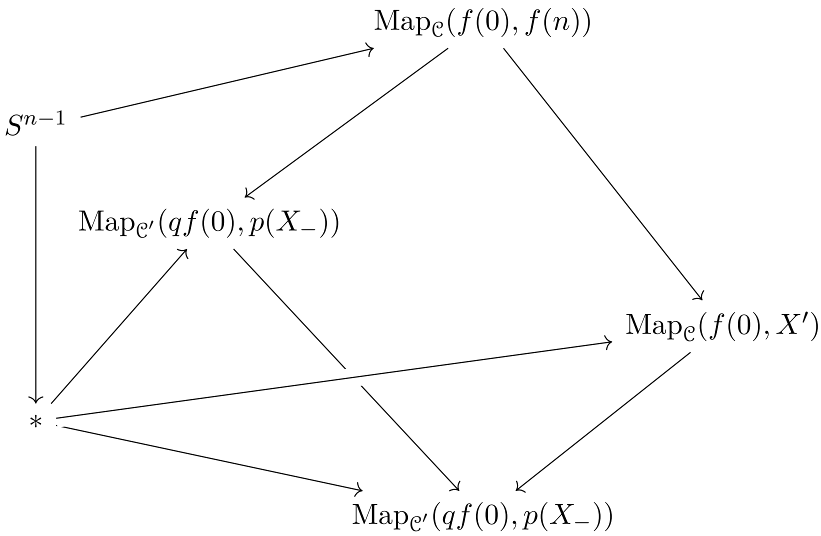
1. *It is S-invariant;*
2. *It is essentially surjective;*
3. *Given any diagram ($n \geq 1$), such that $p|_{f_-}$ is an equivalence, there exists extension $\tilde{p}$ with $\tilde{p}|_{f_-}$ in S:*

$$\begin{array}{ccc} E_n & \longrightarrow & \mathcal{C} \\ \big\downarrow & \overset{\tilde{p}}{\nearrow} & \big\downarrow q \\ D_n^- & \underset{p}{\longrightarrow} & \mathcal{C}' \end{array}$$

The most subtle part is to understand the extension property (3), and we can give a alternative formulation parallel to the discussion of Ore condition in the previous section. Any diagram $f : E_n \to \mathcal{C}$ provides a map $S^{n-1} \to \mathrm{Map}_{\mathcal{C}}(f(0), f(n))$ and a square as in condition (3) gives a commutative square in $\mathcal{S}$:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(f(0), f(n)) \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & \mathrm{Map}_{\mathcal{C}'}(qf(0), p(X_-)) \end{array}$$

To find a diagonal map is equivalent to find an object $X' \in \mathcal{C}$ such that $q(X') \simeq p(X_-)$ (we need the equivalence of categorical fibrations between ∞-categories and iso-fibrations in this step, cf. .....), and a morphism $f(n) \to X' \in S$ and an extension of the above square to the following polyhedron:



Take the fibers in the above square, we obtain another equivalent formulation:

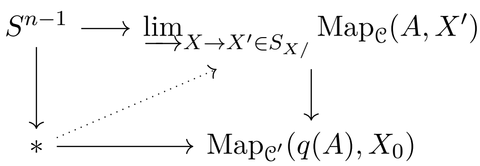
Corollary 1.5.2. *The condition* (3) *for S is equivalent to that, given any objects $A, X \in \mathcal{C}$, object $X_0 \in \mathcal{C}'$, morphism $f : q(A) \to X_0 \in \mathcal{C}_{/X_0}$, equivalence $e : q(X) \to X_0 \in \mathcal{C}_{/X_0}$ and map $S^{n-1} \to \mathrm{Map}_{\mathcal{C}_{/q(X)}}(f, e)$, there exists $s : X \to X' \in S$ such that the composition $S^{n-1} \to \mathrm{Map}_{\mathcal{C}_{/X_0}}(f, e) \to \mathrm{Map}_{\mathcal{C}_{/X_0}}(f, q(s)^{-1}e)$ is nullhomotopic.*

Since $S_{X/}$ is filtered for all $X \in \mathcal{C}$, the previous condition is equivalent to that $\varinjlim_{s:X \to X' \in S_{X/}} \mathrm{Map}_{\mathcal{C}_{/X_0}}(f, q(s)^{-1}e) \simeq *$ holds.

Proof of Theorem 1.5.1. Using the S-invariance of q, given any equivalence $q(X) \to X_0$, we have the following canonical map:

$$\varinjlim_{X \to X' \in S_{X/}} \mathrm{Map}_{\mathcal{C}}(A, X') \to \mathrm{Map}_{\mathcal{C}'}(q(A), X_0)$$

According to Whitehead theorem, this is an equivalence if and only if the following extension problems always have solutions in $\mathcal{S}$:



Taking the fibers, and using the fact that filtered colimits commute with finite limits, we observe that the fibers are always contractible is equivalent to the last condition in Corollary 1.5.2.

For the necessity of these three conditions, (1) and (2) follows from definition, and by the discussion of the first section in this chapter, we have equivalence:

$$\mathrm{Map}_{S^{-1}\mathcal{C}}(A, X) \simeq \mathrm{colim}_{X \to X' \in S_{X/}} \mathrm{Map}_{\mathcal{C}}(A, X')$$

Hence, the condition (3) is equivalent to the full-faithfulness of $S^{-1}\mathcal{C} \to \mathcal{C}'$. For the sufficiency, we can apply Theorem 1.1.5 to the previous discussion to show S admits right calculus of fractions. Then (2) and (3) guarantees the essential surjectivity and fully-faithfulness of $S^{-1}\mathcal{C} \to \mathcal{C}'$, respectively. □

Remark 1.5.3. *Actually the conditions* (1) *and* (3) *of Theorem 1.5.1 are enough to show that S admits right calculus of fractions and the induced functor $S^{-1}\mathcal{C} \to \mathcal{C}'$ is fully faithful.*

1.6 Pullback of Localization Functor

Given a pullback square of small ∞-categories, and a class of morphisms $S \subseteq \mathcal{C}$:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{C} \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \mathcal{B} & \longrightarrow & \mathcal{D} \end{array}$$

Let $\bar{S}$ be the class of morphisms $f \in \mathcal{A}$, such that the image of f in $\mathcal{C}$ lying in S and the image of f in $\mathcal{B}$ being equivalence.

Theorem 1.6.1. *In the above pullback square, assume that q is a categorical fibration and a localization functor at the class S which admits right calculus of fractions. Then $\bar{S}$ admits right calculus of fractions and q' is a localization functor at $\bar{S}$.*

1.7 Maximal ∞-Groupoid of Localization

2 Colimit Indexed by Contractible ∞-Categories

3 Waldhausen ∞-Category

4 Quotient of Stable ∞-Categories

4.1 Quotient by Calculus of Fractions

Definition 4.1.1. *Let $\mathcal{D}$ be a small stable ∞-category and $\mathcal{N} \subseteq \mathcal{D}$ a stable subcategory. The class $S_{\mathcal{N}}$ of equivalences mod $\mathcal{N}$ consists of morphisms in $\mathcal{D}$ of which cofibers lie in $\mathcal{N}$.*

Remark 4.1.2. *If the subcategory $\mathcal{N}$ is clear in the context, we will write S for $S_{\mathcal{N}}$ for the sake of simplicity.*

Definition 4.1.3. *An exact functor $f : \mathcal{D} \to \mathcal{D}'$ between small stable ∞-categories is $\mathcal{N}$-acyclic if it maps objects of $\mathcal{N}$ to zero objects. We will write $\mathrm{Fun}^{\mathrm{ex}}_{\mathcal{N}}(\mathcal{D}, \mathcal{D}')$ for the full subcategory of $\mathcal{N}$-acyclic exact functors. A quotient of $\mathcal{D}$ by $\mathcal{N}$ is an $\mathcal{N}$-acyclic exact functor $q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ such that it induces equivalence of categories for any $\mathcal{D}'$:*

$$q^* : \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}/\mathcal{N}, \mathcal{D}') \simeq \mathrm{Fun}^{\mathrm{ex}}_{\mathcal{N}}(\mathcal{D}, \mathcal{D}')$$

The quotient of a small stable ∞-category $\mathcal{D}$ by any stable subcategory $\mathcal{N}$ is always exist since it is a homotopy pushout in $\mathrm{Stab}\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty}$:

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{D}/\mathcal{N} \end{array}$$

Here 0 denotes the trivial category since it is zero object in $\mathrm{Stab}\mathcal{C}\mathrm{at}^{\mathrm{ex}}_{\infty}$. Sometimes such squares are called short exact sequences of stable ∞-categories.

Lemma 4.1.4. *An exact functor is $\mathcal{N}$-acyclic if and only if it is $S_{\mathcal{N}}$-invariant.*

Proof. By the definition. □

Lemma 4.1.5. *The class $S_{\mathcal{N}}$ contains all equivalences and is closed under composition. The ∞-category $S_{X/}$ is filtered.*

Proof. The first two claims can be deduced from the fact that $\mathcal{N}$ contains zero objects and is closed under extension. The category $S_{X/}$ is equivalent to $\mathcal{N}_{/X}$ by taking fibers. Since $\mathcal{N}$ is stable, it admits finite colimits and hence $\mathcal{N}_{/X}$ also admits finite colimits. It follows that $\mathcal{N}_{/X}$ is filtered. □

The main theorem of this section is the following.

Theorem 4.1.6. *The class $S_{\mathcal{N}}$ admits calculus of left and right fractions.*

Proof. By applying opposite category, we only need to prove that $S_{\mathcal{N}}$ admits calculus of right fractions. According to Theorem 1.1.5, by the definition of $S_{\mathcal{N}}$, we have to prove that $\mathrm{Map}_{\mathrm{ind}\text{-}\mathcal{D}}(N, LX) \simeq *$ for any $N \in \mathcal{N}$. Using the stability of $\mathcal{D}$ and the fact that filtered colimits commute with π_0, we only need to prove the following filtered colimit of abelian groups is trivial:

$$\varinjlim_{X \to X' \in S_{X/}} \pi_0 \mathrm{Map}_{\mathcal{D}}(N, X') \simeq 0$$

The following argument justifies our claim. Given any $u \in \pi_0 \mathrm{Map}_{\mathcal{D}}(N, X')$, namely a morphism $u : N \to X'$, we can take the cofiber of u to get a morphism $X' \to X''$ in $S_{\mathcal{N}}$. The composition $X \to X' \to X''$ is in $S_{\mathcal{N}}$ and hence u vanishes in $\pi_0 \mathrm{Map}_{\mathcal{D}}(N, X'')$. □

Corollary 4.1.7. *The ∞-category $S_{\mathcal{N}}^{-1}\mathcal{D}$ is stable and the localization functor $q : \mathcal{D} \to S_{\mathcal{N}}^{-1}\mathcal{D}$ is exact.*

Proof. Since $S_{\mathcal{N}}$ admits left and calculus of right fractions, by Theorem 1.2.2, q is exact and $S_{\mathcal{N}}^{-1}\mathcal{D}$ admits zero objects, finite limits and colimits. The essential surjectivity and exactness of q implies that $\Sigma\Omega X \simeq X \simeq \Omega\Sigma X$ for any $X \in S_{\mathcal{N}}^{-1}\mathcal{D}$ since it holds for $\mathcal{D}$. □

We can now establish the following theorem.

Theorem 4.1.8. *We have canonical equivalence $\mathcal{D}/\mathcal{N} \simeq S_{\mathcal{N}}^{-1}\mathcal{D}$.*

Proof. By Theorem 1.2.3 and the above results. □

As a corollary, we can show that, the homotopy category of quotient (as triangulated category) is the Verdier quotient of homotopy categories.

Corollary 4.1.9. *We have canonical equivalence of triangulated categories:*

$$h\mathcal{D}/h\mathcal{N} \simeq h(\mathcal{D}/\mathcal{N})$$

The left-hand-side is the Verdier quotient of triangulated categories.

Proof. We have a canonical triangulated functor $h\mathcal{D}/h\mathcal{N} \to h(\mathcal{D}/\mathcal{N})$ By universal property. However, The Verdier quotient $h\mathcal{D}/h\mathcal{N}$ can be seen as localization of $h\mathcal{D}$ at $hS_{\mathcal{N}}$ and since taking homotopy categories commutes with localization, both side is equivalent to $hS_{\mathcal{N}}^{-1}h\mathcal{D}$. □

4.2 Homological Functors

Let $\mathcal{D}$ be a stable ∞-category and $\mathcal{A}$ be an abelian category. A homological functor $h : \mathcal{D} \to \mathcal{A}$ is an additive functor that sends cofiber sequences to exact sequences. If we set $h_n(X) \simeq h(\Sigma^{-n}X)$, we obtain naturally long exact sequences for a cofiber sequence $X \to Y \to Z$ by extending it in both ends:

$$\cdots \to h_{n+1}(Z) \to h_n(X) \to h_n(Y) \to h_n(Z) \to h_{n-1}(X) \to \dots$$

Notice that, a homological functor is $\mathcal{N}$-acyclic for some saturated subcategory if and only if it is $S_{\mathcal{N}}$-invariant by long exact sequence.

Theorem 4.2.1. *Let $\mathcal{D}$ be a small stable ∞-category and $\mathcal{N} \subseteq \mathcal{D}$ a saturated subcategory. Given a homological functor $h : \mathcal{D} \to \mathcal{A}$ that is $\mathcal{N}$-acyclic, the functor induced by universal property $\tilde{h} : \mathcal{D}/\mathcal{N} \to \mathcal{A}$ is homological.*

Proof. Let the quotient functor be $q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$. The point is, any morphism in $\mathcal{D}/\mathcal{N}$ is equivalent to some $q(f)$ for some morphism f in $\mathcal{D}$. It follows that any cofiber sequence in $\mathcal{D}/\mathcal{N}$ is equivalent to the image under q of certain cofiber sequence in $\mathcal{D}$. So the theorem follows from $h \simeq \tilde{h} \circ q$ and h is homological. □

Even if the homological functor is not $\mathcal{N}$-acyclic, we can still extend it to the quotient while keep it being homological under certain circumstances.

Theorem 4.2.2. *Let $\mathcal{D}$ be a small stable ∞-category, $\mathcal{N} \subseteq \mathcal{D}$ a saturated subcategory and $\mathcal{A}$ an abelian category that admits filtered colimits and in which taking filtered colimits is exact. Given a homological functor $h : \mathcal{D} \to \mathcal{A}$, its left Kan extension along quotient functor $\tilde{h} : \mathcal{D}/\mathcal{N} \to \mathcal{A}$ is homological.*

4.3 Compatibility with Small (co-)Limits

Theorem 4.3.1. *Let $\mathcal{D}$ be a small stable ∞-category and $\mathcal{N} \subseteq \mathcal{D}$ a stable subcategory. Assume that $\mathcal{D}$ admits coproducts indexed by small set I and $\mathcal{N}$ is closed under such coproducts. Then the quotient $\mathcal{D}/\mathcal{N}$ admits coproducts indexed by I and the quotient functor $q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ preserves such coproducts.*

Corollary 4.3.2. *Let $\mathcal{D}$ be a small stable ∞-category and $\mathcal{N} \subseteq \mathcal{D}$ a stable subcategory. Given a regular cardinal κ, assume that $\mathcal{D}$ admits κ-small colimits and $\mathcal{N}$ is closed under such colimits. Then the quotient $\mathcal{D}/\mathcal{N}$ admits κ-small colimits and the quotient functor $q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ preserves such colimits.*

4.4 Quotient of Stable Subcategories

Let $\mathcal{D}$ be a small stable ∞-category and $\mathcal{D}_0, \mathcal{N} \subseteq \mathcal{D}$ stable subcategories. We will use $\mathcal{D}_0 \cap \mathcal{N}$ to denote the full stable subcategory of $\mathcal{D}_0$ with objects equivalent to some objects in $\mathcal{N}$. By the universal property of quotients, we have a canonical functor $\mathcal{D}_0 \,/\, \mathcal{D}_0 \cap \mathcal{N} \to \mathcal{D}/\mathcal{N}$ which is not necessarily fully faithful in general.

Theorem 4.4.1. *Suppose that one of the following conditions holds:*

1. *Any morphism $N \to X$ in $\mathcal{D}$ with $X \in \mathcal{D}_0$ and $N \in \mathcal{N}$ can be extended to a square with $N' \in \mathcal{D}_0 \cap \mathcal{N}$, $X' \in \mathcal{D}_0$ and $s \in S_{\mathcal{N}}$:*

$$\begin{array}{ccc} N & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle s} \\ N' & \longrightarrow & X' \end{array}$$

2. *Any morphism $X \to N$ in $\mathcal{D}$ with $X \in \mathcal{D}_0$ and $N \in \mathcal{N}$ can be extended to a square with $N' \in \mathcal{D}_0 \cap \mathcal{N}$, $X' \in \mathcal{D}_0$ and $s \in S_{\mathcal{N}}$:*

$$\begin{array}{ccc} X & \longrightarrow & N \\ \downarrow{\scriptstyle s} & & \downarrow \\ X' & \longrightarrow & N' \end{array}$$

Then the canonical functor $\mathcal{D}_0 \,/\, \mathcal{D}_0 \cap \mathcal{N} \to \mathcal{D}/\mathcal{N}$ is fully faithful.

Proof. We will apply Theorem 1.3.1 to the case when (i) holds and the case for (ii) can be deduced by using opposite category. □

Theorem 4.4.2. *Suppose that one of the following conditions holds:*

1. *For any $X \in \mathcal{D}$, there exists $s : X \to X_0$ with $s \in S_{\mathcal{N}}$ and $X_0 \in \mathcal{D}_0$;*

2. *For any $X \in \mathcal{D}$, there exists $s : X_0 \to X$ with $s \in S_{\mathcal{N}}$ and $X_0 \in \mathcal{D}_0$;*

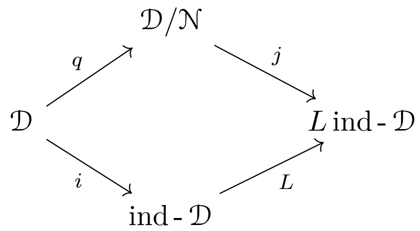
Then the canonical functor $\mathcal{D}_0 \,/\, \mathcal{D}_0 \cap \mathcal{N} \to \mathcal{D}/\mathcal{N}$ is equivalence.

4.5 Kan Extension along Quotient Functors

Let $\mathcal{D}$ be a small stable ∞-category and $\mathcal{N} \subseteq \mathcal{D}$ a stable subcategory. The quotient functor is denoted as $q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$. Given any exact functor $f : \mathcal{D} \to \mathcal{D}'$, we will study the Kan extension of f along q.

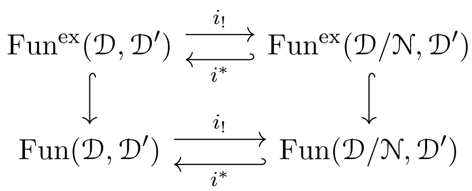
Theorem 4.5.1. *The left or right Kan extensions (if exist) of exact functors between stable ∞-categories along any quotient functor are still exact.*

Proof. By using opposite category, we only need to prove the case of left Kan extension. We can enlarge $\mathcal{D}'$ as in HTT Lemma 5.3.5.7. and without loss of generality, one can assume that $\mathcal{D}'$ admits small colimits. The construction of calculus of fractions shows that $\mathcal{D}/\mathcal{N}$ is a full subcategory of $L\,\mathrm{ind}\text{-}\mathcal{D}$, that the functor L is the localization of $\mathrm{ind}\text{-}\mathcal{D}$ at $S_{\mathcal{N}}$ and we have a commutative square in which i and j are fully faithful:



The fully-faithfulness of j shows that the restriction of the left Kan extension of $f : \mathcal{D} \to \mathcal{D}'$ along $Li \simeq jq$ to $\mathcal{D}/\mathcal{N}$ is the left Kan extension along q. Our proposition follows if we can show that left Kan extension of exact functor along i and L are still exact. The claim for i follows from HTT Proposition 5.5.1.9. And the claim for L holds because L admits a right adjoint, which is the inclusion of $L\,\mathrm{ind}\text{-}\mathcal{D}$ into $\mathrm{ind}\text{-}\mathcal{D}$. Hence left Kan extension along L is just the composition with this inclusion. ☐

We have adjoint functors whenever left Kan extensions always exist.



Given exact multi-functor $f : \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_n \to \mathcal{D}$ and quotients $q_i : \mathcal{D}_i \to \mathcal{D}_i \,/\, \mathcal{N}_i$, the left Kan extension of f along $q_1 \times q_2 \times \cdots \times q_n$ is still exact if it exists.

Theorem 4.5.2. *The left or right Kan extensions (if exist) of exact multi-functors between stable ∞-categories along product of quotient functors are still exact.*

Proof. Similar to the proof of the previous theorem, we can use opposite category and enlarge $\mathcal{D}'$ if necessary, we only need to prove the case of left Kan extension and $\mathcal{D}'$ admits small colimits. We have equivalence (the notion $\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n), \mathcal{D}')$ represents for the full subcategory of $\mathrm{Fun}(\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_n, \mathcal{D}')$ that consists of exact multi-functors):

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n), \mathcal{D}') \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{n-1}), \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}_n, \mathcal{D}'))$$

Using the previous adjoint functors, we can construct inductively the left adjoint of restriction q^*:

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n), \mathcal{D}') \underset{q^*}{\overset{q_!}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{ex}}((\mathcal{D}_1 / \mathcal{N}_1, \mathcal{D}_2 / \mathcal{N}_2, \dots, \mathcal{D}_n / \mathcal{N}_n), \mathcal{D}')$$

Notice that, at each stage the left adjoint is the restriction of the left adjoint defined on $\mathrm{Fun}(\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_n, \mathcal{D}')$, hence it is the left Kan extension. □

Remark 4.5.3. *Another way to prove Theoren 4.5.2 is using the tensor product of small stable ∞-categories to reduce it to Theorem 4.5.1 since we have equivalence:*

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n), \mathcal{D}') \simeq \mathrm{Fun}(\mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \cdots \otimes \mathcal{D}_n, \mathcal{D}')$$

It follows that, Kan extensions of f along $\Pi_i q_i$ is equivalent to Kan extensions along $\otimes_i q_i$:

$$\otimes_i q_i : \mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \cdots \otimes \mathcal{D}_n \to \mathcal{D}_1 / \mathcal{N}_1 \otimes \mathcal{D}_2 / \mathcal{N}_2 \otimes \cdots \otimes \mathcal{D}_n / \mathcal{N}_n$$

4.6 Quotient of Locally Small Stable ∞-Categories

Definition 4.6.1. *A sequence of small (resp. presentable) stable ∞-categories as follows is called exact, if i is fully faithful, qi maps everything to zero object, and q induces fully faithful embedding $\mathcal{D} / \mathcal{D}'$ into $\mathcal{D}''$ which becomes equivalence after idempotent completion (resp. becomes equivalence):*

$$\mathcal{D}' \xrightarrow{i} \mathcal{D}' \xrightarrow{q} \mathcal{D}''$$

Theorem 4.6.2. *A sequence of small stable ∞-categories is exact if and only if the corresponding sequence of* ind*-objects is exact sequence of presentable stable ∞-categories :*

$$\operatorname{ind-}\mathcal{D}' \xrightarrow{i} \operatorname{ind-}\mathcal{D}' \xrightarrow{q} \operatorname{ind-}\mathcal{D}''$$

Corollary 4.6.3. *Given a compactly generated stable ∞-category $\mathcal{D}$,*

$$\mathcal{D}' \xrightarrow{i} \mathcal{D}' \xrightarrow{q} \mathcal{D}''$$

4.7 Pullbacks and Lax Pullbacks

4.8 Quotient of Diagram ∞-Categories

5 Homotopy ∞-Category of Additive Categories

The aim of this chapter is to establish the universal property for the bounded homotopy ∞-category of a small additive category. We begin with some definitions.

Definition 5.0.1. *Let $\mathcal{A}$ be an additive category and $\mathcal{C}$ an ∞-category. An additive functor $f : \mathcal{A} \to \mathcal{C}$ is a functor that preserves initial objects and finite coproducts.*

We will use $\mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{C})$ to denote the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ comprised of additive functors.

The aim of this chapter is to construct what we call the bounded homotopy ∞-category $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ of a small abelian category $\mathcal{A}$ (also called ∞-category of bounded chain complexes, free stable ∞-category generated by $\mathcal{A}$ using finite limits and colimits or even longer names, whatever), to compare our description with the classical homotopy category of chain complexes, that gives categorical alternative of the classical construction, and then to establish the following universal property of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$:

Theorem 5.0.2. *Let $\mathcal{A}$ be a small additive category. The inclusion from $\mathcal{A}$ to its bounded homotopy ∞-category is the initial additive functor towards stable ∞-categories:*

$$i : \mathcal{A} \to \mathbf{K}^{\mathrm{b}}(\mathcal{A})$$

Namely, it induces equivalence of categories for any stable ∞-category $\mathcal{D}$:

$$i^* : \mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{D})$$

Hence one can say that $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ is freely generated by $\mathcal{A}$. This theorem (or just some defintions with comparisons) will be established through the end of this chapter.

5.1 Freely Finite-Generated ∞-Category

We will use $\mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{C}')$ to denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ comprised by right exact functors.

Definition 5.1.1. *Let $\mathcal{A}$ be a small additive category. The free ∞-category generated by $\mathcal{A}$ using finite colimits is an additive functor $j : \mathcal{A} \to \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ such that it induces equivalence of categories for any ∞-category $\mathcal{C}$ that admits finite colimits:*

$$j^* : \mathrm{Fun}^{\mathrm{rex}}(\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A}), \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{C})$$

Let us use $\mathrm{Fun}^{\times}(\mathcal{C}, \mathcal{D})$ to denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ comprised of functors that preserve terminal objects and finite products. Notice that $\mathrm{Fun}^{\times}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{add}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$. Using HTT Proposition 5.3.6.2, the construction $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ exists and can be identified as the smallest subcategory of $\mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$ (this category is a reflective localization of $\mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$) that contains $\mathcal{A}$ and is closed under finite colimits. Also the same proposition guarantees that $j : \mathcal{A} \to \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ is fully faithful and $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ admits zero objects (we call such ∞-categories pointed). The limits in $\mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathcal{S})$ can be computed termwisely but the colimits gain no clear description from this general proposition. However, we have the following lemma that will give us a helping hand, which shows in our case colimits are also computed termwisely.

Lemma 5.1.2. *Given a small additive category $\mathcal{A}$, we have the following canonical equivalence:*

$$\Omega^{\infty} : \mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{S}\mathrm{p}_{\geq 0}) \simeq \mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{S})$$

Proof. First, we check that $\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{S})$ is a so-called additive ∞-category (cf. SAG Definition C.1.5.1). It is deduced from the fact that $\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{S})$ is equivalent to $\mathcal{P}_{\Sigma}(\mathcal{A}^{\mathrm{op}})$ (cf. HTT Definition 5.5.8.8) and hence we can use SAG Lemma C.1.5.8. Let us denote the ∞-categories of commutative monoid objects of $\mathcal{C}$ as $\mathrm{Comm}(\mathcal{C})$, and the subcategory of group-like ones as $\mathrm{CommGrp}(\mathcal{C})$. A combination of HA Propositions 2.4.2.5 and 2.4.3.8 (cf. SAG Remark C.1.5.3) shows that:

$$\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{S}) \simeq \mathrm{Comm}(\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{S}))$$

The right-hand side is equivalent to $\mathrm{Fun}^{\times}(\mathcal{A}, \mathrm{Comm}(\mathcal{S}))$ since the forgetful functor $\mathrm{Comm}(\mathcal{S}) \to \mathcal{S}$ reflects finite products. We are left to prove that $\mathrm{Fun}^{\times}(\mathcal{A}, \mathrm{Comm}(\mathcal{S})) \simeq \mathrm{Fun}^{\times}(\mathcal{A}, \mathrm{CommGrp}(\mathcal{S}))$ since $\mathcal{S}\mathrm{p}_{\geq 0} \simeq \mathrm{CommGrp}(\mathcal{S})$. The point is, the image of any product-preserving functor $f : \mathcal{A} \to \mathrm{Comm}(\mathcal{S})$ is automatically group-like because for any object $A \in \mathcal{A}$, the morphism $-id_A$ induces inverse map $f(A) \to f(A)$. $\square$

Remark 5.1.3. *The* ∞*-category* $\mathrm{Fun}^\times(\mathcal{A}, \mathcal{S}\mathrm{p}_{\geq 0})$ *is pre-stable in the sense of SAG Definition C.1.2.1.*

The previous discussion leads to the following theorem, which gives a partial description of mapping spaces in $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$:

Theorem 5.1.4. *the following properties hold for* $j : \mathcal{A} \to \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$*:*

1. *The suspension functor* $\Sigma : \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A}) \to \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ *is fully faithful;*
2. *The functor* j *is fully faithful and moreover for any* $n \neq 0$ *and objects* $A, A' \in \mathcal{A}$ *we have:*

$$\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})}(A, \Sigma^n A') \simeq 0$$

Proof. The fully-faithfulness of j has been shown in the beginning of this section. The claim (1) follows from the fact that $\Sigma : \mathcal{S}\mathrm{p}_{\geq 0} \to \mathcal{S}\mathrm{p}_{\geq 0}$ is fully faithful. To prove the vanishing of π_0, we can embed $\mathcal{S}\mathrm{p}_{\geq 0}$ into $\mathcal{S}\mathrm{p}$, which preserves small colimits, and it follows that we can regard $\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}}, \mathcal{S}\mathrm{p}_{\geq 0})$ as a full subcategory closed under colimits of the stable ∞-category $\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}}, \mathcal{S}\mathrm{p})$. Therefore we have equivalence (we use $\mathcal{M}\mathrm{ap}$ to denote the mapping spectrum):

$$\operatorname{Map}_{\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})}(A, \Sigma^n A') \simeq \Sigma^n \mathcal{M}\mathrm{ap}_{\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}}, \mathcal{S}\mathrm{p})}(A, A')$$

However, $\mathcal{M}\mathrm{ap}_{\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}}, \mathcal{S}\mathrm{p})}(A, A')$ is connective by definition. So the π_0 of the right-hand-side vanishes when $n > 0$. The case $n < 0$ is implied by the fully-faithfullness of j. □

Remark 5.1.5. *There is a dual notion of free* ∞*-category generated using finite limits, which is an additive functor* $j : \mathcal{A} \to \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ *such that it induces equivalence of categories for any* ∞*-category* $\mathcal{C}$ *that admits finite limits:*

$$j^* : \mathrm{Fun}^{\mathrm{lex}}(\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A}), \mathcal{C}) \simeq \mathrm{Fun}^\times(\mathcal{A}, \mathcal{C})$$

Here $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{C}')$ *denotes the full subcategory of* $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ *comprised of left exact functors. All results in this section has dual form for* $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ *by taking opposite categories.*

5.2 Freely Finite-Generated Stable ∞-Category

In this section study the stable ∞-category $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$, which corresponds to the classical bounded chain complexes of an additive category $\mathcal{A}$.

Definition 5.2.1. *Let $\mathcal{A}$ be a small additive category. The free stable ∞-category generated by $\mathcal{A}$ (using finite limits/colimits) is an additive functor $i : \mathcal{A} \to \mathbf{K}^{\mathrm{b}}(\mathcal{A})$ such that it induces equivalence of categories for any stable ∞-category $\mathcal{D}$:*

$$i^* : \mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{D})$$

This construction always exists. The reason is that the functor $\mathcal{D} \mapsto \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{D})$ preserves small limits and is accessible (it preserves κ-filtered colimits if $\mathcal{A}$ is κ-small). And since $\mathrm{Stab}\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}}$ is presentable, we can use representable functor theorem to find $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$.

The main result in this section is to provide a criterion to recognize $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$, in order to show in subsequent sections, that the classical construction using chain complexes satisfies the previous universal property. Before that, we will show first that $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ is the Spanier-Whitehead ∞-category of $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$, which we introduced in the last section.

Definition 5.2.2. *Let $\mathcal{C}$ be a small pointed ∞-category that admits finite colimits. The Spanier-Whitehead ∞-category $\mathrm{SW}(\mathcal{C})$ is the direct limit iof the following diagram in $\mathcal{C}\mathrm{at}_{\infty}$:*

$$\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \dots$$

We have a canonical functor $\Sigma^{\infty} : \mathcal{C} \to \mathrm{SW}(\mathcal{C})$.

The following lemma is from SAG Proposition C.1.1.7:

Lemma 5.2.3. *Let $\mathcal{C}$ be a small pointed ∞-category that admits finite colimits. Then the Spanier-Whitehead ∞-category $\mathrm{SW}(\mathcal{C})$ is stable, and the canonical functor $\Sigma^{\infty} : \mathcal{C} \to \mathrm{SW}(\mathcal{C})$ induces equivalence of categories for any stable ∞-category $\mathcal{D}$:*

$$(\Sigma^{\infty})^* : \mathrm{Fun}^{\mathrm{ex}}(\mathrm{SW}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{D})$$

Moreover, any object of $\mathrm{SW}(\mathcal{C})$ is equivalent to iterated desuspensions of objects of the form $\Sigma^{\infty} X$, and hence the smallest stable subcategory that contains the essential image of Σ^{∞} is equivalent to $\mathrm{SW}(\mathcal{C})$ itself.

Corollary 5.2.4. *We have canonical equivalence $\mathbf{K}^{\mathrm{b}}(\mathcal{A}) \simeq \mathrm{SW}(\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A}))$.*

Proof. Combine the universal property of $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and Lemma 5.2.3. □

Remark 5.2.5. *Since $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ is a full subcategory of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$, the pushouts in $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ are automatically pullbacks.*

The Spanier-Whitehead construction leads to the following computation of mapping spaces of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$:

Lemma 5.2.6. *The functor $i : \mathcal{A} \to \mathbf{K}^{\mathrm{b}}(\mathcal{A})$ is fully faithful and moreover for any $n \neq 0$ and objects $A, A' \in \mathcal{A}$ we have:*

$$\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(A, \Sigma^n A') \simeq 0$$

Proof. By the definition, the mapping spaces of Spanier-Whitehead construction has the following description (we identify $\Sigma^\infty X$ with X by abuse of notation):

$$\operatorname{Map}_{\mathrm{SW}(\mathcal{C})}(X, \Sigma^n Y) \simeq \varinjlim_{p>-n} \operatorname{Map}_{\mathcal{C}}(\Sigma^p X, \Sigma^{p+n} Y)$$

It follows that we have equivalences:

$$\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(A, \Sigma^n A') \simeq \varinjlim_{p>-n} \pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})}(\Sigma^p A, \Sigma^{p+n} A')$$

Therefore Theorem 5.1.4 concludes the proof. □

Here comes our criterion for $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$.

Theorem 5.2.7. *Let $\mathcal{A}$ be an additive category. Given an additive functor $f : \mathcal{A} \to \mathcal{D}$ towards a stable ∞-category $\mathcal{D}$, the canonical exact functor $\tilde{f} : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ is an equivalence if and only if the functor f satisfies the following properties:*

1. *The smallest stable subcategory of $\mathcal{D}$ that contains the essential image of f is equivalent to $\mathcal{D}$ itself;*
2. *The functor f is fully faithful and moreover for any $n \neq 0$ and objects $A, A' \in \mathcal{A}$ we have:*

$$\pi_0 \operatorname{Map}_{\mathcal{D}}(f(A), \Sigma^n f(A')) \simeq 0$$

Proof. We have shown that $i : \mathcal{A} \to \mathbf{K}^{\mathrm{b}}(\mathcal{A})$ satisfies these properties. For the sake of convenience, we will regard $\mathcal{A}$ as full subcategories of both $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ and $\mathcal{D}$. The property (2) is equivalent to that, $\tilde{f}$ induces the following equivalence of mapping spectra:

$$\mathcal{M}\mathrm{ap}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(A, A') \simeq \mathcal{M}\mathrm{ap}_{\mathcal{D}}(A, A')$$

Notice that, after fixing any object $A \in \mathcal{A}$, the full subcategory of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ comprised of those objects X for which $\mathcal{M}\mathrm{ap}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(A, X) \simeq \mathcal{M}\mathrm{ap}_{\mathcal{D}}(A, X)$ holds is a stable subcategory (since $\tilde{f}$ is exact) of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ that contains $\mathcal{A}$. It follows from property (1) of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ that this subcategory is the whole $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$, and a similar argument shows that $\tilde{f}$ is actually a fully faithful functor. Now we can regard $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ as a full stable subcategory of $\mathcal{D}$ and of course it contains $\mathcal{A}$. So property (1) to $\mathcal{D}$ guarantees $\tilde{f}$ is an equivalence of categories. □

Remark 5.2.8. *The proof shows that condition (2) is equivalent to that $\tilde{f}$ is fully faithful.*

We can regard $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ (or dually $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$) as full subcategory of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$, and it follows immediately from the construction the following characterizations of these two subcategories:

Theorem 5.2.9. *Let $\mathcal{A}$ be an additive category. The following holds:*

1. *The ∞-category $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ is the smallest full subcategory of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ that contains $\mathcal{A}$ and is closed under finite colimits;*

2. *The ∞-category $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ is the smallest full subcategory of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ that contains $\mathcal{A}$ and is closed under finite limits.*

The following generalization of Lemma 5.2.6 will be used later.

Lemma 5.2.10. *The following holds:*

1. *Given any objects $X \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and $Y \in \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$, for $n > 0$ we have:*

$$\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{-n} Y) \simeq 0$$

2. *Given any objects $X \in \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ and $Y \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$, for $n > 0$ we have:*

$$\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{n} Y) \simeq 0$$

Proof. Fixing an object $A \in \mathcal{A}$, let us use $\mathcal{D}$ to denote the full subcategory of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ comprised of objects Y that $\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(A, \Sigma^{-n} Y) \simeq 0$ for all $n > 0$. The long exact sequence of π_n implies that $\mathcal{D}$ is closed under finite limits, and Lemma 5.2.6 shows that $\mathcal{D}$ contains $\mathcal{A}$. It follow from Theorem 5.2.9 that $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ is generated by $\mathcal{A}$ using finite limits, and therefore $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A}) \subseteq \mathcal{D}$. A similar argument concludes the proof of (1). The proof of (2) is similiar. □

5.3 Finite $\mathcal{A}$-Decompositions

In this section, we prove that objects in $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ can be endowed with structure similar to finite CW-complexes, which is the reason that why such objects can be represented by chain complexes.

Definition 5.3.1. *Let $\mathcal{A}$ be an additive category. A finite $\mathcal{A}$-complex $X_\bullet$ of length n in $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ is a co-tower:*

$$X_{\leq 0} \to X_{\leq 1} \to X_{\leq 2} \to \dots X_{\leq n-1} \to X_{\leq n}$$

Together with cofiber sequences such that $X_p \in \mathcal{A}$ for each p:

$$\Sigma^p X_{p+1} \to X_{\leq p} \to X_{\leq p+1}$$

Notice that we have $X_0 \simeq X_{\leq 0}$. A morphism $f_\bullet : X_\bullet \to Y_\bullet$ between $\mathcal{A}$-complexes, is a series of morphisms $f_p : X_p \to Y_p$ and $f_{\leq p} : X_{\leq p} \to Y_{\leq p}$ that constitutes morphisms between co-towers and cofiber sequences. Given an object $X \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$, a finite $\mathcal{A}$-decomposition is a finite $\mathcal{A}$-complex $X_\bullet$ with length n and an equivalence $X \simeq X_{\leq n}$.

Remark 5.3.2. *In general, $\mathcal{A}$-decomposition is not unique.*

The main result of this section is the following.

Theorem 5.3.3. *Given an additive category $\mathcal{A}$, any object $X \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ admits finite $\mathcal{A}$-decomposition.*

Before prove that, we need some lemmas first.

Lemma 5.3.4. *Given a finite $\mathcal{A}$-complex $X_\bullet$, the cofiber of $X_{\leq p} \to X_{\leq p+i}$ for $i \geq 0$ is contained in $\Sigma^{p+1}\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$.*

Proof. We do induction on i. The case $i = 0$ is trivial. Assume that $\operatorname{cofib}(X_{\leq p} \to X_{\leq p+i}) \in \Sigma^{p+1}\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$, the 2-simplex $X_{\leq p} \to X_{\leq p+i} \to X_{\leq p+i+1}$ induces cofiber sequence in $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$:

$$\Sigma^{-1}\operatorname{cofib}(X_{\leq p+i} \to X_{\leq p+i+1}) \to \operatorname{cofib}(X_{\leq p} \to X_{\leq p+i}) \to \operatorname{cofib}(X_{\leq p} \to X_{\leq p+i+1})$$

Notice that $\Sigma^{-1}\operatorname{cofib}(X_{\leq p+i} \to X_{\leq p+i+1}) \simeq \Sigma^{p+i}X_{p+i+1} \in \Sigma^{p+1}\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$, and by induction assumption, we deduce that $\operatorname{cofib}(X_{\leq p} \to X_{\leq p+i}) \in \Sigma^{p+1}\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$. □

Lemma 5.3.5. *Let $\mathcal{A}$ be an additive category. Given objects $X, Y \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and their $\mathcal{A}$-decompositions $X_\bullet, Y_\bullet$, any morphism $f : X \to Y$ can be extended to morphism between $\mathcal{A}$-complexes $f_\bullet : X_\bullet \to Y_\bullet$.*

Proof. We will construct $f_p : X_p \to Y_p$ and $f_{\leq p} : X_{\leq p} \to Y_{\leq p}$ inductively. Given $f_{\leq p} : X_{\leq p} \to Y_{\leq p}$, to construct $f_{\leq p+1}$, we need to solve the following extension problem:

$$\begin{array}{ccccc} X_{\leq p} & & \longrightarrow & & X \\ \downarrow & \searrow & & \nearrow & \downarrow \\ & & X_{\leq p+1} & & \\ & & \vdots & & \\ Y_{\leq p} & & \longrightarrow & & Y \\ & \searrow & \downarrow & \nearrow & \\ & & Y_{\leq p+1} & & \end{array}$$

It is equivalent to the following extension problem:

$$\begin{array}{ccc} X_{\leq p} & \longrightarrow & X_{\leq p+1} \\ \downarrow & \nearrow & \downarrow \\ Y_{\leq p+1} & \longrightarrow & Y \end{array}$$

By obstruction theory in $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$, we only need to show that the induced morphism on cofibers $\mathrm{cofib}(X_{\leq p} \to X_{\leq p+1}) \to \mathrm{cofib}(Y_{\leq p+1} \to Y)$ is zero. Notice that $\mathrm{cofib}(X_{\leq p} \to X_{\leq p+1}) \simeq \Sigma^{p+1} X_{p+1}$, and hence we can use Lemma 5.2.10 and the previous lemma to show that it is always zero.

After we construct $f_{\leq p+1}$, we can take the fibers of the rows in the following square to construct f_{p+1}:

$$\begin{array}{ccc} X_{\leq p} & \longrightarrow & X_{\leq p+1} \\ \downarrow & & \downarrow \\ Y_{\leq p} & \longrightarrow & Y_{\leq p+1} \end{array}$$

□

Lemma 5.3.6. *Let $\mathcal{A}$ be an additive category. Given objects $X, Y \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$, their $\mathcal{A}$-decompositions $X_\bullet, Y_\bullet$, and morphism $f : X \to Y$. The cofiber $Z \simeq \mathrm{cofib}(X \to Y)$ admits $\mathcal{A}$-decompositions $Z_\bullet$ such that $Z_p \simeq X_{p-1} \oplus Y_p$.*

Proof. We use the previous lemma to extend morphism $X \to Y$ to $X_\bullet \to Y_\bullet$ and claim that the following co-tower meets our requirements:

$$\mathrm{cofib}(X_{\leq 0} \to Y_{\leq 1}) \to \mathrm{cofib}(X_{\leq 1} \to Y_{\leq 2}) \to \cdots \to \mathrm{cofib}(X_{\leq p} \to Y_{\leq p+1}) \to \cdots$$

It is clear that when p is big enough, we have $\mathrm{cofib}(X_{\leq p} \to Y_{\leq p+1}) \simeq \mathrm{cofib}(X \to Y) \simeq Z$. Let F_p denote the fiber of $\mathrm{cofib}(X_{\leq p-1} \to Y_{\leq p}) \to$

$\mathrm{cofib}(X_{\leq p} \to Y_{\leq p+1})$, we have cofiber sequence:

$$\Sigma^{p-1} X_p \to \Sigma^p Y_{p+1} \to F_p$$

By Lemma 5.2.10, the morphism $\Sigma^{p-1} X_p \to \Sigma^p Y_{p+1}$ is zero, hence we have equivalence $F_p \simeq \Sigma^p (X_p \oplus Y_{p+1})$ and cofiber sequence:

$$\Sigma^p (X_p \oplus Y_{p+1}) \to \mathrm{cofib}(X_{\leq p-1} \to Y_{\leq p}) \to \mathrm{cofib}(X_{\leq p} \to Y_{\leq p+1})$$

This justifies our claim. □

Now we can prove the main result.

Proof of Theorem 5.3.3. Let $\mathcal{D}$ be the full subcategory of $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ spanned by objects satisfying this theorem. We claim that $\mathcal{D}$ is closed under finite colimits. It is clear that $\mathcal{D}$ admits zero objects and closed under finite coproducts and the previous lemma shows that it is closed under taking cofibers. To finish the proof, notice that $\mathcal{A} \subseteq \mathcal{D}$ and hence we have $\mathcal{D} \simeq \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$. □

We can use Lemma 5.3.6 to show the pre-stability of $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ (cf. SAG Definition C.1.2.1 for the definition of pre-stability).

Corollary 5.3.7. *The ∞-category $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ is pre-stable.*

Proof. For any morphism $X \to \Sigma Y$ such that $X, Y \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$, Lemma 5.3.6 shows that the its cofiber Z admits $\mathcal{A}$-decomposition $Z_\bullet$ with $Z_{\leq 0} \simeq 0$ and hence $Z \in \mathbf{K}^{\mathrm{b}}_{\geq 1}(\mathcal{A})$ or $\Sigma^{-1} Z \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$. □

Corollary 5.3.8. *Given a small abelian category $\mathcal{A}$, and an object $X \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$. There exists a cofiber sequence with $A \in \mathcal{A}$ and $X' \in \mathbf{K}^{\mathrm{b}}_{\geq 1}(\mathcal{A})$:*

$$A \to X \to X'$$

Proof. Take a finite $\mathcal{A}$-decomposition of X and let A be X_0. □

Remark 5.3.9. *The results in this section are direct consequences of the results in the next section. However, our proof here has no dependence on any strict models of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$.*

In the end of this section, we reformulate our main result in a relative form for later use.

Theorem 5.3.10. *Given a morphism $X \to X' \in \mathbf{K}^{\mathrm{b}}(\mathcal{A})$ such that jj*

5.4 Bounded Chain Complexes

In this section, we will show that $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ can be constructed by chain complexes, as in the classical theory. Let $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A})$ (resp. $\mathrm{Ch}(\mathcal{A})$) denote the differential graded category of bounded (resp. unbounded) chain complexes of $\mathcal{A}$, and N_{dg} the differential graded nerve construction (cf. HA Construction 1.3.1.6). We need the following lemma from HA Corollary 1.3.2.16 to compute homotopy pushouts of chain complexes.

Lemma 5.4.1. *Let $\mathcal{A}$ be an additive category and suppose we have a pushout square in $\mathrm{Ch}(\mathcal{A})$ (seen as 1-category):*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

If f is termwisely split, namely each $X_n \to Y_n$ admits left inverse, then this is a pushout square in $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$.

As direct corollaries, the mapping cone of any map between chain complexes gives out cofiber sequence:

$$X \xrightarrow{f} Y \longrightarrow \mathrm{C}\, f$$

For a chain complex $X \simeq \cdots \to X_{n+1} \to X_n \to X_{n-1} \to \ldots$, let $\mathrm{t}_{\geq n} X$ and $\mathrm{t}_{\leq n} X$ denote the truncations of chain complexes $\cdots \to X_{n+1} \to X_n$ and $X_n \to X_{n-1} \to \ldots$ respectively. We have cofiber sequence:

$$\mathrm{t}_{\leq n} X \longrightarrow X \longrightarrow \mathrm{t}_{\geq n+1} X$$

The following is also cofiber sequence:

$$\Sigma^n X_{n+1} \longrightarrow \mathrm{t}_{\leq n} X \longrightarrow \mathrm{t}_{\leq n+1} X$$

Corollary 5.4.2. *We have canonical equivalence $\mathbf{K}^{\mathrm{b}}(\mathcal{A}) \simeq \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^{\mathrm{b}}(\mathcal{A}))$.*

Proof. We need to check the two conditions given in Theorem 5.2.7. According to HTT Proposition 1.3.2.10, $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ is stable. The above discussion shows that $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^{\mathrm{b}}(\mathcal{A}))$ is a full subcategory closed under taking cofiber sequences and desuspensions, and is generated by $\mathcal{A}$ using these two operations. Hence the condition (1) holds. The condition (2) follows from the representation of mapping spaces of $\mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}(\mathcal{A}))$ as Hom-complexes (cf. HA Proposition 1.3.2.23). □

Remark 5.4.3. *It can be deduced from equivalence $\mathbf{K}^{\mathrm{b}}(\mathcal{A}) \simeq \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^{\mathrm{b}}(\mathcal{A}))$ and the previous characterizations that that $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A}) \simeq \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^{\mathrm{b}}_{\geq 0}(\mathcal{A}))$ and $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A}) \simeq \mathrm{N}_{\mathrm{dg}}(\mathrm{Ch}^{\mathrm{b}}_{\leq 0}(\mathcal{A}))$. Here $\mathrm{Ch}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ (resp. $\mathrm{Ch}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$) is the dg-category of non-negatively (resp. non-positively) homological-indexed and bounded chain complexes of $\mathcal{A}$.*

5.5 Additive Multi-Functors

The universal property of $\mathbf{K}^{\mathrm{b}}$ has natural generalizations to multi-functors, namely functors with many variables as $f : \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n \to \mathcal{C}$. The category of multi-functors will be denoted as $\mathrm{Fun}((\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n), \mathcal{C})$.

Definition 5.5.1. *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be additive categories and $\mathcal{C}$ an ∞-category. An additive multi-functors a functor $f : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{C}$ that preserves initial objects and finite coproducts at each variable.*

We will use $\mathrm{Fun}^{\mathrm{add}}((\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n), \mathcal{C})$ to denote the full subcategory of $\mathrm{Fun}((\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n), \mathcal{C})$ comprised of additive multi-functors.

Theorem 5.5.2. *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be small additive categories and $\mathcal{D}$ be a stable ∞-category. The restriction along $\mathcal{A}_n \to \mathbf{K}^{\mathrm{b}}(\mathcal{A}_n)$ induces equivalence of ∞-categories:*

$$\mathrm{Fun}^{\mathrm{add}}((\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathbf{K}^{\mathrm{b}}(\mathcal{A}_1), \mathbf{K}^{\mathrm{b}}(\mathcal{A}_2), \ldots, \mathbf{K}^{\mathrm{b}}(\mathcal{A}_n)), \mathcal{D})$$

Proof. We have equivalence:

$$\mathrm{Fun}^{\mathrm{add}}((\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{add}}((\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n-1}), \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}_n, \mathcal{D}))$$

Similar equivalences also hold for $\mathrm{Fun}^{\mathrm{ex}}$. So we can do induction the prove our proposition, begin with the universal property of $\mathbf{K}^{\mathrm{b}}$:

$$\begin{aligned}
\mathrm{Fun}^{\mathrm{add}}((\mathcal{A}_1, \mathcal{A}_2), \mathcal{D}) &\simeq \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}_1, \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}_2, \mathcal{D})) \\
&\simeq \mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}_1), \mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}_2), \mathcal{D}) \\
&\simeq \mathrm{Fun}^{\mathrm{ex}}((\mathbf{K}^{\mathrm{b}}(\mathcal{A}_1), \mathbf{K}^{\mathrm{b}}(\mathcal{A}_2)), \mathcal{D})
\end{aligned}$$

□

Remark 5.5.3. *The tensor product of small stable ∞-categories (cf. [Lurie]) allows us to reformulate Theorem 5.5.2 as:*

$$\mathrm{Fun}^{\mathrm{add}}((\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}_1) \otimes \mathbf{K}^{\mathrm{b}}(\mathcal{A}_2) \otimes \cdots \otimes \mathbf{K}^{\mathrm{b}}(\mathcal{A}_n), \mathcal{D})$$

5.6 Adjoining Unbounded Resolutions

In this section, we introduce some constructions that will help use to extend bounded homotopy ∞-category $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ to not necessarily bounded versions $\mathbf{K}^{-}(\mathcal{A})$, $\mathbf{K}^{+}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})$.

Let $\mathcal{D}$ be a stable ∞-category with a full subcategory $\mathcal{D}_{\geq 0}$ closed under direct sums and suspensions. We set $\mathcal{D}_{\geq n} \simeq \Sigma^n \mathcal{D}_{\geq 0}$. A highly connective cotower with respect to $\mathcal{D}_{\geq 0}$ is a diagram $X_\bullet : \mathrm{N}(\mathbb{Z}_{\geq 0}) \to \mathcal{D}$ satisfying the property that, for any integer $m \geq 0$, there exists $n_0 \geq 0$ such that for any $n \geq n_0$, we have $\mathrm{cofib}(X_n \to X_{n+1}) \in \mathcal{D}_{\geq m}$.

$$X_0 \to X_1 \to X_2 \to \cdots \to X_n \to \dots$$

These highly connective cotowers may not have colimits in $\mathcal{D}$, and we can adjoint such colimits freely to $\mathcal{D}$, the result ∞-category will be denoted by $\mathcal{D}^{-}$. Dually, one can consider full subcategory $\mathcal{D}_{\geq 0}$ closed under direct sums and de-suspensions and we can adjoint limits of highly connective towers $\mathrm{N}(\mathbb{Z}_{\geq 0})^{\mathrm{op}} \to \mathcal{D}$ freely to get $\mathcal{D}^{+}$.

Remark 5.6.1. *Using the ∞-categorical Dold-Kan correspondence (cf. [HA Section 1.2.4]), the condition $\mathcal{D}$ admits colimits of highly connective cotowers with respect to $\mathcal{D}_{\geq 0}$ is equivalent to that it admits geometric realizations of simplicial object $X_\bullet : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathcal{D}$ with the property that all terms $X_i \in \mathcal{D}_{\geq n}$ for some fixed integer n.*

To construct $\mathcal{D}^{-}$, one can follow the ideas of [HTT Section 5.3.6]. However, the construction given there cannot be applied directly and we have to make some modifications. The $\infty-$category $\mathcal{D}^{-}$ can be identified as the full subcategory of ind - $\mathcal{D}$ that consists of highly connective cotowers (seen as ind-objects of $\mathcal{D}$). However, even the stability of $\mathcal{D}^{-}$ is not obvious and we need several propositions to assure that our definition is reasonable.

Proposition 5.6.2. *The ∞-category $\mathcal{D}^{-}$ is full stable subcategory of* ind - $\mathcal{D}$.

By Yoneda lemma, the canonical functor $\mathcal{D} \to \mathcal{D}^{-}$ is exact and fully-faithful and hence we can regard $\mathcal{D}_{\geq 0}$ as full subcategory of $\mathcal{D}^{-}$. We will use $\mathcal{D}^{-}_{\geq 0}$ to denote the subcategory of $\mathcal{D}$ consisting of highly connective cotowers with each terms $X_n \in \mathcal{D}^{-}$. The subcategory $\mathcal{D}^{-}_{\geq 0}$ is closed under direct sums and suspensions and contains $\mathcal{D}_{\geq 0}$.

Proposition 5.6.3. *The ∞-category $\mathcal{D}^{-}$ admits colimits of highly connective cotowers with respect to $\mathcal{D}^{-}_{\geq 0}$.*

Now we can describe the universal property of the construction $\mathcal{D}^{-}$. Let $\mathcal{D}$ also be a stable ∞-category with a full subcategory $\mathcal{D}_{\geq 0}$ closed under direct sums and suspensions, and $\mathcal{D}$ admits colimits of highly connective cotowers with respect to $\mathcal{D}'_{\geq 0}$.

Proposition 5.6.4. *The inclusion functor $i : \mathcal{D} \to \mathcal{D}^-$ induced equivalence:*

$$\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D}, \mathcal{D}_{\geq 0}), (\mathcal{D}', \mathcal{D}'_{\geq 0})) \simeq \mathrm{Fun}^{\mathrm{ex},-}((\mathcal{D}^-, \mathcal{D}_{\geq 0}), (\mathcal{D}', \mathcal{D}'_{\geq 0}))$$

The left-hand-side consists of exact functors that sends $\mathcal{D}_{\geq 0}$ to $\mathcal{D}'_{\geq 0}$, and the right-hand-side moreover preserves colimits of highly connective cotowers with respect to $\mathcal{D}_{\geq 0}$.

5.7 Homotopy ∞-Category

In this section, we study $\mathbf{K}^-(\mathcal{A})$, $\mathbf{K}^+(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})$ based on our previous construction of $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$.

Definition 5.7.1. *Given a stable ∞-category $\mathcal{D}$ with a full subcategory $\mathcal{D}_{\geq 0}$ closed under direct sums and suspensions, we set $\mathcal{D}_{\geq n} \simeq \Sigma^n \mathcal{D}_{\geq 0}$. A highly connective tower with respect to $\mathcal{D}_{\geq 0}$ is a diagram $X_\bullet : \mathrm{N}(\mathbb{Z}_{\geq 0}) \to \mathcal{D}$ satisfying the property that, for any integer $m \geq 0$, there exists $n_0 \geq 0$ such that for any $n \geq n_0$, we have* $\mathrm{cofib}(X_n \to X_{n+1}) \in \mathcal{D}_{\geq m}$.

Remark 5.7.2. *In fact, for any $n' \geq n \geq n_0$, we have* $\mathrm{cofib}(X_n \to X_{n'}) \in \mathcal{D}_{\geq m}$.

Definition 5.7.3. *Given a stable ∞-category $\mathcal{D}$ with a full subcategory $\mathcal{D}_{\geq 0}$ closed under direct sums and suspensions, the stable ∞-category $\mathcal{D}^-$ by freely adjoining highly connective towers with respect to $\mathcal{D}_{\geq 0}$ to $\mathcal{D}$ is the free*

6 Bounded Derived ∞-Category

The aim of this chapter is to establish the universal property for the bounded derived ∞-category of a small exact category. We begin with some definitions.

Definition 6.0.1. *Let $\mathcal{E}$ be an exact category and $\mathcal{C}$ an ∞-category. We call a functor $f : \mathcal{E} \to \mathcal{C}$ a δ-functor if it is additive and sends short exact sequences to cofiber sequences.*

We will use $\mathrm{Fun}^{\delta}(\mathcal{E}, \mathcal{C})$ to denote the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{C})$ comprised of δ-functors.

The aim of this chapter is to construct what we call the bounded derived ∞-category $\mathbf{D}^{\mathrm{b}}(\mathcal{E})$ of a small exact category $\mathcal{E}$ with equivalence of triangulated categories $h\,\mathbf{D}^{\mathrm{b}}(\mathcal{E}) \simeq \mathrm{D}^{\mathrm{b}}(\mathcal{E})$(the latter one is the classical derived category), that generalizes the classical construction of derived category by localization, and then to establish the following universal property of $\mathbf{D}^{\mathrm{b}}(\mathcal{E})$:

Theorem 6.0.2. *Let $\mathcal{E}$ be a small exact category. The inclusion from $\mathcal{E}$ to its bounded derived ∞-category is the universal δ-functor towards stable ∞-categories:*

$$i : \mathcal{E} \to \mathbf{D}^{\mathrm{b}}(\mathcal{E})$$

Namely, it induces equivalence of categories for any stable ∞-category $\mathcal{D}$:

$$i^{*} : \mathrm{Fun}^{\mathrm{ex}}(\mathbf{D}^{\mathrm{b}}(\mathcal{E}), \mathcal{D}) \simeq \mathrm{Fun}^{\delta}(\mathcal{E}, \mathcal{D})$$

Hence one can say that $\mathbf{D}^{\mathrm{b}}(\mathcal{E})$ is freely generated by $\mathcal{E}$ modulo the relations given by short exact sequences. This theorem will be established as a concatenation of several universal properties of a series of constructions in the subsequent sections (cf. the proof next to Corollary ...).

6.1 Quasi-Isomorphisms

The bounded chain complexes ∞-category $\mathbf{K}^{\mathrm{b}}(\mathcal{E})$ for an exact category $\mathcal{E}$ is naturally equipped with a full stable subcategory $\mathbf{N}^{\mathrm{b}}(\mathcal{E})$, which is the smallest stable subcategory contains $\mathrm{cofib}(X \to Y) \to Z$

We call an object $X \in \mathbf{K}^{\mathrm{b}}(\mathcal{A})$ is acyclic if $\mathrm{H}_n(X) \simeq 0$ for all n and a morphism $f : X \to Y$ is a quasi-isomorphism if the induced morphisms $f_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ are isomorphisms for all n. The full subcategory of acyclic objects will be denoted as $\mathbf{Ac}^{\mathrm{b}}(\mathcal{A})$ and the class of quasi-isomorphisms as $\mathcal{Q}$. Using the long exact sequences, we see that $\mathbf{N}^{\mathrm{b}}(\mathcal{A})$ is saturated and $\mathcal{Q} \simeq S_{\mathbf{N}^{\mathrm{b}}(\mathcal{A})}$ (cf. Definition 4.1.1). As in the classical case, we use localization at quasi-isomorphisms to define derived category.

Definition 6.1.1. *Let $\mathcal{A}$ be an abelian category. The bounded derived ∞-category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is $\mathcal{Q}^{-1}\,\mathbf{K}^{\mathrm{b}}(\mathcal{A}) \simeq \mathbf{K}^{\mathrm{b}}(\mathcal{A})/\,\mathbf{N}^{\mathrm{b}}(\mathcal{A})$.*

Let $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ denotes the classical derived category (as triangulated category). Since $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is stable, its homotopy category can be endowed with a canonical triangulated structure. By comparing with the definition of $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$, using the fact that localization commutes with taking homotopy categories, we deduce that $h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A}) \simeq \mathrm{D}^{\mathrm{b}}(\mathcal{A})$ as triangulated categories.

The remaining of this section is devoted to study the interrelation of derived category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ and short exact sequences in $\mathcal{A}$, and finally give a proof of Theorem 7.0.2.

Lemma 6.1.2. *Let $\mathcal{A}$ be an abelian category. Given an exact functor $f : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ towards a stable ∞-category $\mathcal{D}$, such that it sends short exact sequences in $\mathcal{A}$ to cofiber sequences. Then f sends short exact sequences of bounded chain complexes to cofiber sequences.*

Proof. Given short exact sequence of bounded complexes $0 \to X \to Y \to Z \to 0$, for convenience, assume that their negative terms all vanish. The proof is to do induction on their truncations. The starting points is $\mathrm{t}_{\leq 0}\,X \to \mathrm{t}_{\leq 0}\,Y \to \mathrm{t}_{\leq 0}\,Z$, which is a short exact sequence in $\mathcal{A}$, so f sends it to cofiber sequence. For the inductive step, we have the following diagram:

$$\begin{array}{ccccc}
\Sigma^n X_{n+1} & \longrightarrow & \mathrm{t}_{\leq n}\,X & \longrightarrow & \mathrm{t}_{\leq n+1}\,X \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^n Y_{n+1} & \longrightarrow & \mathrm{t}_{\leq n}\,Y & \longrightarrow & \mathrm{t}_{\leq n+1}\,Y \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^n Z_{n+1} & \longrightarrow & \mathrm{t}_{\leq n}\,Z & \longrightarrow & \mathrm{t}_{\leq n+1}\,Z
\end{array}$$

The rows are all cofiber sequence and f sends the left column to cofiber sequence since it is suspensions of short exact sequence in $\mathcal{A}$ and f sends

the middle column to cofiber sequence by inductive assumption. Therefore f sends the right column to cofiber sequence. The induction ends at some n because all chain complexes involved here are bounded. □

Theorem 6.1.3. *Let $\mathcal{A}$ be an abelian category. Given an exact functor $f : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ towards a stable ∞-category $\mathcal{D}$, the following conditions for f are equivalent:*

1. *It sends acyclic objects to zero objects;*
2. *It sends quasi-isomorphisms to equivalences;*
3. *It sends short exact sequences in $\mathcal{A}$ to cofiber sequences.*

Proof. The equivalence of (1) and (2) is direct consequence of the definition. For the case (2) $\Rightarrow$ (3), given short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, the canonical morphism $\mathrm{cofib}(X \to Y) \to Z$ is quasi-isomorphism, so if f sends this morphism to equivalence, it will send that exact sequence to cofiber sequence. For the case (3) $\Rightarrow$ (1), given any bounded acyclic complex X as follows (after using (de-)suspension to make X_n at index n), we will do induction on the length n to prove $f(X)$ is zero object.

$$X_{n+1} \to X_n \to \cdots \to X_2 \to X_1 \to X_0$$

The case $n = 0$ is trivial, and in the case $n = 1$, we have $X_1 \simeq X_0$ and hence $X \simeq \mathrm{cofib}(X_1 \simeq X_0) \simeq 0$. For the inductive step, we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccc}
X_{n+1} & \xrightarrow{id} & X_{n+1} & \longrightarrow & 0 & & \\
{\scriptstyle id}\downarrow & & \downarrow & & \downarrow & & \\
X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & {\scriptstyle id}\downarrow & & \\
0 & \longrightarrow & X_n/X_{n+1} & \longrightarrow & X_{n-1} & \longrightarrow & \dots
\end{array}$$

The rows form a short exact sequence of bounded chain complexes:

$$0 \to \Sigma^n \mathrm{cofib}(id_{X_{n+1}}) \to X \to X/\Sigma^n X_{n+1} \to 0$$

So by Lemma 7.1.2, f sents it to cofiber sequence, and the left and right terms become zero objects since we have $\mathrm{cofib}(id_{X_{n+1}}) \simeq 0$ and the inductive assumption. It follows that $f(X) \simeq 0$. □

Corollary 6.1.4. *The quotient functor $q : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ sends short exact sequences of $\mathcal{A}$ to cofiber sequences, and it induces a fully faithful functor for any stable ∞-categories $\mathcal{D}$:*

$$q^* : \mathrm{Fun}^{\mathrm{ex}}(\mathbf{D}^{\mathrm{b}}(\mathcal{A}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}), \mathcal{D})$$

The essential image of q^* consists of exact functors that sends short exact sequences of $\mathcal{A}$ to cofiber sequences.

Proof. By the exactness of q and equivalences in Theorem 7.1.3. □

We are ready to establish the universal property of bounded derived ∞-category, which is promised in the very beginning of this chapter.

Proof of Theorem 7.0.2. We can factorize the functor $i : \mathcal{A} \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ as:

$$\mathcal{A} \to \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A}) \to \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$$

The theorem follows from combining the universal properties of each functor, provided respectively in Definition 5.1.1, Definition 5.2.1 and Corollary 7.1.4. □

6.2 Quasi-Isomorphisms for Abelian Categories

The bounded chain complexes ∞-category $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ for an abelian category $\mathcal{A}$ is naturally equipped with homological functors $\mathrm{H}_n : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{A}$ simply by taking the n^{th} homology object of a chain complex. We call an object $X \in \mathbf{K}^{\mathrm{b}}(\mathcal{A})$ is acyclic if $\mathrm{H}_n(X) \simeq 0$ for all n and a morphism $f : X \to Y$ is a quasi-isomorphism if the induced morphisms $f_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ are isomorphisms for all n. The full subcategory of acyclic objects will be denoted as $\mathbf{N}^{\mathrm{b}}(\mathcal{A})$ and the class of quasi-isomorphisms as $\mathcal{Q}$. Using the long exact sequences, we see that $\mathbf{N}^{\mathrm{b}}(\mathcal{A})$ is saturated and $\mathcal{Q} \simeq S_{\mathbf{N}^{\mathrm{b}}(\mathcal{A})}$ (cf. Definition 4.1.1). As in the classical case, we use localization at quasi-isomorphisms to define derived category.

Definition 6.2.1. *Let $\mathcal{A}$ be an abelian category. The bounded derived ∞-category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is $\mathcal{Q}^{-1}\,\mathbf{K}^{\mathrm{b}}(\mathcal{A}) \simeq \mathbf{K}^{\mathrm{b}}(\mathcal{A})/\,\mathbf{N}^{\mathrm{b}}(\mathcal{A})$.*

Let $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ denotes the classical derived category (as triangulated category). Since $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is stable, its homotopy category can be endowed with a canonical triangulated structure. By comparing with the definition of $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$, using the fact that localization commutes with taking homotopy categories, we deduce that $h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A}) \simeq \mathrm{D}^{\mathrm{b}}(\mathcal{A})$ as triangulated categories.

The remaining of this section is devoted to study the interrelation of derived category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ and short exact sequences in $\mathcal{A}$, and finally give a proof of Theorem 7.0.2.

Lemma 6.2.2. *Let $\mathcal{A}$ be an abelian category. Given an exact functor $f : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ towards a stable ∞-category $\mathcal{D}$, such that it sends short exact sequences in $\mathcal{A}$ to cofiber sequences. Then f sends short exact sequences of bounded chain complexes to cofiber sequences.*

Proof. Given short exact sequence of bounded complexes $0 \to X \to Y \to Z \to 0$, for convenience, assume that their negative terms all vanish. The proof is to do induction on their truncations. The starting points is $\mathrm{t}_{\leq 0}\,X \to \mathrm{t}_{\leq 0}\,Y \to \mathrm{t}_{\leq 0}\,Z$, which is a short exact sequence in $\mathcal{A}$, so f sends it to cofiber sequence. For the inductive step, we have the following diagram:

$$\begin{array}{ccccc}
\Sigma^n X_{n+1} & \longrightarrow & \mathrm{t}_{\leq n}\,X & \longrightarrow & \mathrm{t}_{\leq n+1}\,X \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^n Y_{n+1} & \longrightarrow & \mathrm{t}_{\leq n}\,Y & \longrightarrow & \mathrm{t}_{\leq n+1}\,Y \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^n Z_{n+1} & \longrightarrow & \mathrm{t}_{\leq n}\,Z & \longrightarrow & \mathrm{t}_{\leq n+1}\,Z
\end{array}$$

The rows are all cofiber sequence and f sends the left column to cofiber sequence since it is suspensions of short exact sequence in $\mathcal{A}$ and f sends

the middle column to cofiber sequence by inductive assumption. Therefore f sends the right column to cofiber sequence. The induction ends at some n because all chain complexes involved here are bounded. □

Theorem 6.2.3. *Let $\mathcal{A}$ be an abelian category. Given an exact functor $f : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ towards a stable ∞-category $\mathcal{D}$, the following conditions for f are equivalent:*

1. *It sends acyclic objects to zero objects;*
2. *It sends quasi-isomorphisms to equivalences;*
3. *It sends short exact sequences in $\mathcal{A}$ to cofiber sequences.*

Proof. The equivalence of (1) and (2) is direct consequence of the definition. For the case (2) ⇒ (3), given short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, the canonical morphism $\mathrm{cofib}(X \to Y) \to Z$ is quasi-isomorphism, so if f sends this morphism to equivalence, it will send that exact sequence to cofiber sequence. For the case (3) ⇒ (1), given any bounded acyclic complex X as follows (after using (de-)suspension to make X_n at index n), we will do induction on the length n to prove $f(X)$ is zero object.

$$X_{n+1} \to X_n \to \cdots \to X_2 \to X_1 \to X_0$$

The case $n = 0$ is trivial, and in the case $n = 1$, we have $X_1 \simeq X_0$ and hence $X \simeq \mathrm{cofib}(X_1 \simeq X_0) \simeq 0$. For the inductive step, we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccc}
X_{n+1} & \xrightarrow{id} & X_{n+1} & \longrightarrow & 0 & & \\
{\scriptstyle id}\downarrow & & \downarrow & & \downarrow & & \\
X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & {\scriptstyle id}\downarrow & & \\
0 & \longrightarrow & X_n/X_{n+1} & \longrightarrow & X_{n-1} & \longrightarrow & \dots
\end{array}$$

The rows form a short exact sequence of bounded chain complexes:

$$0 \to \Sigma^n \,\mathrm{cofib}(id_{X_{n+1}}) \to X \to X/\Sigma^n X_{n+1} \to 0$$

So by Lemma 7.1.2, f sents it to cofiber sequence, and the left and right terms become zero objects since we have $\mathrm{cofib}(id_{X_{n+1}}) \simeq 0$ and the inductive assumption. It follows that $f(X) \simeq 0$. □

Corollary 6.2.4. *The quotient functor $q : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ sends short exact sequences of $\mathcal{A}$ to cofiber sequences, and it induces a fully faithful functor for any stable ∞-categories $\mathcal{D}$:*

$$q^* : \mathrm{Fun}^{\mathrm{ex}}(\mathbf{D}^{\mathrm{b}}(\mathcal{A}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}), \mathcal{D})$$

The essential image of q^* consists of exact functors that sends short exact sequences of $\mathcal{A}$ to cofiber sequences.

Proof. By the exactness of q and equivalences in Theorem 7.1.3. □

We are ready to establish the universal property of bounded derived ∞-category, which is promised in the very beginning of this chapter.

Proof of Theorem 7.0.2. We can factorize the functor $i : \mathcal{A} \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ as:

$$\mathcal{A} \to \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A}) \to \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$$

The theorem follows from combining the universal properties of each functor, provided respectively in Definition 5.1.1, Definition 5.2.1 and Corollary 7.1.4. □

6.3 Delta Multi-Functors

The universal property of $\mathbf{D}^{\mathrm{b}}$ has natural generalizations to multi-functors. This section is parallel to Section 5.5.

Definition 6.3.1. *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be exact categories and $\mathcal{C}$ an ∞-category. An δ-multi-functors a functor $f : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{C}$ that is additive and sends short exact sequences to cofiber sequences at each variable. at each variable.*

We will use $\mathrm{Fun}^{\delta}((\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n), \mathcal{C})$ to denote the full subcategory of $\mathrm{Fun}((\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n), \mathcal{C})$ comprised of δ-multi-functors.

Theorem 6.3.2. *Let $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ be small exact categories and $\mathcal{D}$ be a stable ∞-category. The restriction along $\mathcal{E}_n \to \mathbf{D}^{\mathrm{b}}(\mathcal{E}_n)$ induces equivalence of ∞-categories:*

$$\mathrm{Fun}^{\delta}((\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathbf{D}^{\mathrm{b}}(\mathcal{E}_1), \mathbf{D}^{\mathrm{b}}(\mathcal{E}_2), \ldots, \mathbf{D}^{\mathrm{b}}(\mathcal{E}_n)), \mathcal{D})$$

Proof. Using the induction procedure in the proof of Theorem 5.5.2. □

Remark 6.3.3. *The tensor product of small stable ∞-categories (cf. [Lurie]) allows us to reformulate Theorem 6.3.2 as:*

$$\mathrm{Fun}^{\delta}((\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathbf{D}^{\mathrm{b}}(\mathcal{A}_1) \otimes \mathbf{D}^{\mathrm{b}}(\mathcal{A}_2) \otimes \cdots \otimes \mathbf{D}^{\mathrm{b}}(\mathcal{A}_n), \mathcal{D})$$

6.4 Compatibilty with Certain Categorical Constructions

Theorem 6.4.1. *Given small abelian categories $\mathcal{A}$, we have canonical equivalence:*

$$\mathbf{D}^{\mathrm{b}}(\mathcal{A}^{\mathrm{op}}) \simeq \mathbf{D}^{\mathrm{b}}(\mathcal{A})^{\mathrm{op}}$$

Theorem 6.4.2. *Given small abelian categories $\mathcal{A}_1$ and $\mathcal{A}_2$, we have canonical equivalence:*

$$\mathbf{D}^{\mathrm{b}}(\mathcal{A}_1 \times \mathcal{A}_2) \simeq \mathbf{D}^{\mathrm{b}}(\mathcal{A}_1) \times \mathbf{D}^{\mathrm{b}}(\mathcal{A}_2)$$

Theorem 6.4.3. *Given a filtered diagram indexed by $\mathcal{I}$ of small abelian categories with exact functors between them, we have canonical equivalence:*

$$\mathbf{D}^{\mathrm{b}}(\varinjlim_{i\in\mathcal{I}} \mathcal{A}_i) \simeq \varinjlim_{i\in\mathcal{I}} \mathbf{D}^{\mathrm{b}}(\mathcal{A}_i)$$

Theorem 6.4.4. *Given an small abelian category $\mathcal{A}$ and its full Serre subcategory $\mathcal{B}$ closed under quotients and subobjects, we have canonical equivalence:*

$$\mathbf{D}^{\mathrm{b}}(\mathcal{A}/\mathcal{B}) \simeq \mathbf{D}^{\mathrm{b}}(\mathcal{A})/\,\mathbf{D}^{\mathrm{b}}_{\mathcal{B}}(\mathcal{A})$$

Theorem 6.4.5. *Given a pair of adjoint functors, that both need to be exact, between small abelian categories:*

$$\mathcal{A} \rightleftarrows \mathcal{A}'$$

It induces adjoint functors:

$$\mathbf{D}^{\mathrm{b}}(\mathcal{A}) \rightleftarrows \mathbf{D}^{\mathrm{b}}(\mathcal{A}')$$

We also have some not-so-easy results that will be proved in later chapters.

7 Bounded Derived ∞-Category of Abelian Categories

The aim of this chapter is to establish the universal property for the bounded derived category of a small abelian category. We begin with some definitions.

Definition 7.0.1. *Let $\mathcal{A}$ be an additive category and $\mathcal{C}$ an ∞-category. An additive functor $f : \mathcal{A} \to \mathcal{C}$ is a functor that preserves initial objects and finite coproducts. Moreover assume that $\mathcal{A}$ is an abelian category, we call f a δ-functor if it is additive and sends short exact sequences to cofiber sequences.*

We will use $\mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{C})$ and $\mathrm{Fun}^{\delta}(\mathcal{A}, \mathcal{C})$ to denote the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ comprised of additive functors and δ-functors respectively.

The aim of this chapter is to construct what we call the bounded derived ∞-category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ of a small abelian category $\mathcal{A}$ with equivalence of triangulated categories $h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A}) \simeq \mathrm{D}^{\mathrm{b}}(\mathcal{A})$(the latter one is the classical derived category), that generalizes the classical construction of derived category by localization, and then to establish the following universal property of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$:

Theorem 7.0.2. *Let $\mathcal{A}$ be a small abelian category. The inclusion from $\mathcal{A}$ to its bounded derived ∞-category is the universal δ-functor towards stable ∞-categories:*

$$i : \mathcal{A} \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$$

Namely, it induces equivalence of categories for any stable ∞-category $\mathcal{D}$:

$$i^* : \mathrm{Fun}^{\mathrm{ex}}(\mathbf{D}^{\mathrm{b}}(\mathcal{A}), \mathcal{D}) \simeq \mathrm{Fun}^{\delta}(\mathcal{A}, \mathcal{D})$$

Hence one can say that $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is freely generated by $\mathcal{A}$ modulo the relations given by short exact sequences. This theorem will be established as a concatenation of several universal properties of a series of constructions in the subsequent sections (cf. the proof next to Corollary 7.1.4).

7.1 Quasi-Isomorphisms

The bounded chain complexes ∞-category $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$ for an abelian category $\mathcal{A}$ is naturally equipped with homological functors $\mathrm{H}_n : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{A}$ simply by taking the n^{th} homology object of a chain complex. We call an object $X \in \mathbf{K}^{\mathrm{b}}(\mathcal{A})$ is acyclic if $\mathrm{H}_n(X) \simeq 0$ for all n and a morphism $f : X \to Y$ is a quasi-isomorphism if the induced morphisms $f_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ are isomorphisms for all n. The full subcategory of acyclic objects will be denoted as $\mathbf{N}^{\mathrm{b}}(\mathcal{A})$ $\mathbf{Ac}^{\mathrm{b}}(\mathcal{A})$ and the class of quasi-isomorphisms as $\mathcal{Q}$. Using the long exact sequences, we see that $\mathbf{N}^{\mathrm{b}}(\mathcal{A})$ is saturated and $\mathcal{Q} \simeq S_{\mathbf{N}^{\mathrm{b}}(\mathcal{A})}$ (cf. Definition 4.1.1). As in the classical case, we use localization at quasi-isomorphisms to define derived category.

Definition 7.1.1. *Let $\mathcal{A}$ be an abelian category. The bounded derived ∞-category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is $\mathcal{Q}^{-1}\,\mathbf{K}^{\mathrm{b}}(\mathcal{A}) \simeq \mathbf{K}^{\mathrm{b}}(\mathcal{A})/\,\mathbf{N}^{\mathrm{b}}(\mathcal{A})$.*

Let $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ denotes the classical derived category (as triangulated category). Since $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is stable, its homotopy category can be endowed with a canonical triangulated structure. By comparing with the definition of $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$, using the fact that localization commutes with taking homotopy categories, we deduce that $h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A}) \simeq \mathrm{D}^{\mathrm{b}}(\mathcal{A})$ as triangulated categories.

The remaining of this section is devoted to study the interrelation of derived category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ and short exact sequences in $\mathcal{A}$, and finally give a proof of Theorem 7.0.2.

Lemma 7.1.2. *Let $\mathcal{A}$ be an abelian category. Given an exact functor $f : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ towards a stable ∞-category $\mathcal{D}$, such that it sends short exact sequences in $\mathcal{A}$ to cofiber sequences. Then f sends short exact sequences of bounded chain complexes to cofiber sequences.*

Proof. Given short exact sequence of bounded complexes $0 \to X \to Y \to Z \to 0$, for convenience, assume that their negative terms all vanish. The proof is to do induction on their truncations. The starting points is $\mathrm{t}_{\leq 0}\,X \to \mathrm{t}_{\leq 0}\,Y \to \mathrm{t}_{\leq 0}\,Z$, which is a short exact sequence in $\mathcal{A}$, so f sends it to cofiber sequence. For the inductive step, we have the following diagram:

$$\begin{array}{ccccc} \Sigma^n X_{n+1} & \longrightarrow & \mathrm{t}_{\leq n}\,X & \longrightarrow & \mathrm{t}_{\leq n+1}\,X \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^n Y_{n+1} & \longrightarrow & \mathrm{t}_{\leq n}\,Y & \longrightarrow & \mathrm{t}_{\leq n+1}\,Y \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^n Z_{n+1} & \longrightarrow & \mathrm{t}_{\leq n}\,Z & \longrightarrow & \mathrm{t}_{\leq n+1}\,Z \end{array}$$

The rows are all cofiber sequence and f sends the left column to cofiber sequence since it is suspensions of short exact sequence in $\mathcal{A}$ and f sends

the middle column to cofiber sequence by inductive assumption. Therefore f sends the right column to cofiber sequence. The induction ends at some n because all chain complexes involved here are bounded. □

Theorem 7.1.3. *Let $\mathcal{A}$ be an abelian category. Given an exact functor $f : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ towards a stable ∞-category $\mathcal{D}$, the following conditions for f are equivalent:*

1. *It sends acyclic objects to zero objects;*
2. *It sends quasi-isomorphisms to equivalences;*
3. *It sends short exact sequences in $\mathcal{A}$ to cofiber sequences.*

Proof. The equivalence of (1) and (2) is direct consequence of the definition. For the case (2) $\Rightarrow$ (3), given short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, the canonical morphism $\mathrm{cofib}(X \to Y) \to Z$ is quasi-isomorphism, so if f sends this morphism to equivalence, it will send that exact sequence to cofiber sequence. For the case (3) $\Rightarrow$ (1), given any bounded acyclic complex X as follows (after using (de-)suspension to make X_n at index n), we will do induction on the length n to prove $f(X)$ is zero object.

$$X_{n+1} \to X_n \to \cdots \to X_2 \to X_1 \to X_0$$

The case $n = 0$ is trivial, and in the case $n = 1$, we have $X_1 \simeq X_0$ and hence $X \simeq \mathrm{cofib}(X_1 \simeq X_0) \simeq 0$. For the inductive step, we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccc}
X_{n+1} & \xrightarrow{id} & X_{n+1} & \longrightarrow & 0 & & \\
{\scriptstyle id}\downarrow & & \downarrow & & \downarrow & & \\
X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & {\scriptstyle id}\downarrow & & \\
0 & \longrightarrow & X_n/X_{n+1} & \longrightarrow & X_{n-1} & \longrightarrow & \dots
\end{array}$$

The rows form a short exact sequence of bounded chain complexes:

$$0 \to \Sigma^n \mathrm{cofib}(id_{X_{n+1}}) \to X \to X/\Sigma^n X_{n+1} \to 0$$

So by Lemma 7.1.2, f sents it to cofiber sequence, and the left and right terms become zero objects since we have $\mathrm{cofib}(id_{X_{n+1}}) \simeq 0$ and the inductive assumption. It follows that $f(X) \simeq 0$. □

Corollary 7.1.4. *The quotient functor $q : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ sends short exact sequences of $\mathcal{A}$ to cofiber sequences, and it induces a fully faithful functor for any stable ∞-categories $\mathcal{D}$:*

$$q^* : \mathrm{Fun}^{\mathrm{ex}}(\mathbf{D}^{\mathrm{b}}(\mathcal{A}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}), \mathcal{D})$$

The essential image of q^* consists of exact functors that sends short exact sequences of $\mathcal{A}$ to cofiber sequences.

Proof. By the exactness of q and equivalences in Theorem 7.1.3. □

We are ready to establish the universal property of bounded derived ∞-category, which is promised in the very beginning of this chapter.

Proof of Theorem 7.0.2. We can factorize the functor $i : \mathcal{A} \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ as:

$$\mathcal{A} \to \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A}) \to \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$$

The theorem follows from combining the universal properties of each functor, provided respectively in Definition 5.1.1, Definition 5.2.1 and Corollary 7.1.4. □

7.2 Canonical t-Structure

Given an abelian category $\mathcal{A}$, by Theorem 4.2.2, we have homological functors $\mathrm{H}_n : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{A}$ induced from $\mathbf{K}^{\mathrm{b}}(\mathcal{A})$.

Definition 7.2.1. *Let $\mathcal{A}$ be an abelian category. We say an object $X \in \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is n-connective (resp. n-coconnective) if $\mathrm{H}_p(X) \simeq 0$ for $p < n$ (resp. for $p > n$). The full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ comprised of n-connective (resp. n-coconnective) objects will be denoted as $\mathbf{D}^{\mathrm{b}}_{\geq n}(\mathcal{A})$ (resp. $\mathbf{D}^{\mathrm{b}}_{\leq n}(\mathcal{A})$). We will call 0-connective (resp. 0-coconnective) objects simply as connective (resp. coconnective) objects.*

The category $\mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and $\mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ are actually localization of $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ at quasi-isomorphisms. To prove this, we introduce some constructions first. Given a chain complex X, we have the following diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
\dots & \longrightarrow & X_{n+1} & \longrightarrow & d_{n+1}(X_{n+1}) & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & X_{n+1} & \xrightarrow{d_{n+1}} & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
 & & 0 & \longrightarrow & X_n/d_{n+1}(X_{n+1}) & \longrightarrow & X_{n-1} & \longrightarrow & \dots
\end{array}
$$

The rows form an exact sequence of chain complexes which we will write as:

$$0 \to \mathrm{b}'_{\geq n+1} X \to X \to \mathrm{b}_{\leq n} X \to 0$$

Notice that $\mathrm{b}'_{\geq n} X$ is n-connective and $\mathrm{b}_{\leq n} X$ is n-coconnective. Also the map $\mathrm{b}'_{\geq n} X \to X$ induces isomorphisms on H_p for $p \geq n$ and $X \to \mathrm{b}_{\leq n} X$ induces isomorphisms on H_p for $p \leq n$. Dually one has:

$$
\begin{array}{ccccccccc}
\dots & \longrightarrow & X_{n+1} & \longrightarrow & \ker(d_n) & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \xrightarrow{d_n} & X_{n-1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
 & & 0 & \longrightarrow & X_n/\ker(d_n) & \longrightarrow & X_{n-1} & \longrightarrow & \dots
\end{array}
$$

The corresponding exact sequence will be denoted as:

$$0 \to \mathrm{b}_{\geq n} X \to X \to \mathrm{b}'_{\leq n-1} X \to 0$$

Similar results about connectivity holds for these dual constructions. We also have natural maps $\mathrm{b}_{\geq n} X \to \mathrm{b}'_{\geq n} X$ and $\mathrm{b}_{\leq n} X \to \mathrm{b}'_{\leq n} X$ which are all quasi-isomorphisms.

Theorem 7.2.2. *Let $\mathcal{A}$ be an abelian category. The following holds:*

1. *Let $\mathcal{Q}_{\geq 0} \simeq \mathcal{Q} \cap \mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$. We have equivalence $\mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A}) \simeq \mathcal{Q}^{-1}_{\geq 0}\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$;*

2. *Let $\mathcal{Q}_{\leq 0} \simeq \mathcal{Q} \cap \mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$. We have equivalence $\mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A}) \simeq \mathcal{Q}^{-1}_{\leq 0}\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$.*

Proof. By using opposite category, we only need to prove (2). We have a natural functor $f : \mathcal{Q}^{-1}_{\leq 0}\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A}) \subseteq \mathbf{D}^{\mathrm{b}}(\mathcal{A})$. First, we prove that this functor is fully faithful by applying Theorem 1.3.1. It is enough to show that given $X \in \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ and quasi-isomorphism $X \to X' \in \mathbf{K}^{\mathrm{b}}(\mathcal{A})$, there exists quasi-isomorphism $X' \to X''$ with $X'' \in \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$. The morphism $X \to \mathrm{b}_{\leq 0} X$ meets requirements. The functor f is essentially surjective follows from the fact that, given any coconnective chain complex X, we have the quasi-isomorphism $X \to \mathrm{b}_{\leq 0} X$ (again) such that $\mathrm{b}_{\leq 0} X \in \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$. ☐

In the case of abelian categories, we have the following partial generalizations of Lemma 5.2.6.

Lemma 7.2.3. *For any objects $X \in \mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and $Y \in \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$, the map induced by H_0 is isomorphism:*

$$\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, Y) \simeq \operatorname{Hom}_{\mathcal{A}}(\mathrm{H}_0(X), \mathrm{H}_0(Y))$$

Proof. Using long exact sequences of π_n and Lemma 5.2.10, we obtain isomorphism:

$$\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, Y) \simeq \pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(\mathrm{t}_{\leq 1} X, \mathrm{t}_{\geq -1} Y)$$

So we can assume both X and Y have length 1 and notice that under such assumption we have $X \simeq \operatorname{cofib}(X_1 \xrightarrow{d_1} X_0)$ and $Y \simeq \operatorname{fib}(Y_0 \xrightarrow{d'_0} Y_{-1})$. Now using the long exact sequence of π_n again, we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, Y) & \longrightarrow & \pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X_0, Y) & \longrightarrow & \pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X_1, Y) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, Y_0) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(X_0, Y_0) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(X_1, Y_0) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, Y_{-1}) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(X_0, Y_{-1}) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(X_1, Y_{-1})
\end{array}$$

It follows that:

$$\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, Y) \simeq \operatorname{Hom}_{\mathcal{A}}(\operatorname{coker}(d_1), \ker(d'_0)) \simeq \operatorname{Hom}_{\mathcal{A}}(\mathrm{H}_0(X), \mathrm{H}_0(Y))$$

☐

Lemma 7.2.4. *Let $\mathcal{A}$ be an abelian category. For any objects $X \in \mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and $Y \in \mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$, the map induced by H_0 is isomorphism:*

$$\pi_0 \operatorname{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, Y) \simeq \operatorname{Hom}_{\mathcal{A}}(\mathrm{H}_0(X), \mathrm{H}_0(Y))$$

And the following holds for $n > 0$:

$$\pi_0 \operatorname{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{-n} Y) \simeq 0$$

Proof. The Theorem 7.2.2 allows us to compute these π_0 by the following filtered colimits:

$$\pi_0 \operatorname{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{-n} Y) \simeq \varinjlim_{Y \to Y' \in (\mathcal{Q}_{\leq 0})_{Y/}} \pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^{-n} Y')$$

We can use Lemma 5.2.10 (1) and Lemma 7.2.3 to decide the right-hand-side and the theorem follows immediately. □

A direct consequence is that we can regard $\mathcal{A}$ as full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$.

Corollary 7.2.5. *Given any abelian category $\mathcal{A}$, the functor towards its bounded derived ∞-category $i : \mathcal{A} \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is fully faithful.*

We have the following characterization similar to Theorem 5.2.9:

Theorem 7.2.6. *Let $\mathcal{A}$ be an abelian category. The following holds:*

1. *The ∞-category $\mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ is the smallest full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ that contains $\mathcal{A}$ and is closed under finite colimits;*
2. *The ∞-category $\mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ is the smallest full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ that contains $\mathcal{A}$ and is closed under finite limits.*

Proof. By using opposite category, we only need to prove (1). Let $\mathcal{D}$ denote the smallest full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ that contains $\mathcal{A}$ and is closed under finite colimits. It follow from Theorem 5.2.9 that the essential image of $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ under the quotient functor is contained in $\mathcal{D}$. However, Theorem 7.2.2 shows that the essential image of $\mathbf{K}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ is $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ and $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ itself contains $\mathcal{A}$ and is closed under finite colimits. Therefore $\mathbf{D}^{\mathrm{b}}(\mathcal{A}) \simeq \mathcal{D}$. □

The last result in this section is to equip $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ with a t-structure, of which heart recovers $\mathcal{A}$.

Theorem 7.2.7. *The pair of subcategories $(\mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A}), \mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A}))$ provides a t-structure for $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$, which we call the canonical t-structure. The heart of the canonical t-structure is equivalent to $\mathcal{A}$:*

$$\mathcal{A} \simeq \mathbf{D}^{\mathrm{b}}(\mathcal{A})^{\heartsuit}$$

Moreover, the homological functor H_0 is equivalent to the bi-side truncation $\pi_0 : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})^{\heartsuit}$.

Proof. The only axiom for t-structure left is that for any object $X \in \mathbf{D}^{\mathrm{b}}(\mathcal{A})$, there exists cofiber sequence $X' \to X \to X''$ such that $X' \in \Sigma\,\mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and $X'' \in \mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$. The cofiber sequence $\mathrm{b}'_{\geq 1} X \to X \to \mathrm{b}_{\leq 0} X$ meets the requirements. Definitely, we have $\mathcal{A} \subseteq \mathbf{D}^{\mathrm{b}}(\mathcal{A})^{\heartsuit}$. To prove the inclusion of the other way around, consider the zig-zag $X \to \mathrm{b}_{\leq 0} X \leftarrow \mathrm{b}_{\geq 0}\, \mathrm{b}_{\leq 0} X$. If X is inside the heart, both maps are quasi-isomorphisms and $\mathrm{b}_{\geq 0}\, \mathrm{b}_{\leq 0} X \in \mathcal{A}$. This also shows the last claim. The proof has been done. □

For stable ∞-category $\mathcal{D}$ with t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$, let $\mathcal{D}_{[a,b]}$ be the intersection $\mathcal{D}_{\geq a} \cap \mathcal{D}_{\leq b}$.

Theorem 7.2.8. *The following is equivalent for t-structures:*

1. *The smallest stable subcategory containing $\mathcal{D}^{\heartsuit}$ is $\mathcal{D}$;*

2. *Any object in $\mathcal{D}$ is contained in some $\mathcal{D}_{[a,b]}$.*

Proof. (1) $\Rightarrow$ (2) The union $\cup_{a \leq b} \mathcal{D}_{[a,b]}$ is the smallest stable subcategory containing $\mathcal{D}^{\heartsuit}$ by induction;

(2) $\Rightarrow$ (1) Using the Postnikov tower forf t-structures. □

Notice that bounded t-structure is non-degenerate and its heart cannot be trivial unless the whole category is trivial.

Definition 7.2.9. *We call a t-structure bounded if it satisfies the above equivalent properties.*

Theorem 7.2.10. *The canonical t-structure of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ is bounded.*

Proof. Using (1) of Theorem 5.2.7. □

I do not know whether the following theorem holds for triangulated category. Our proof, though simple, uses the ∞-categorical structures in an essential way.

Theorem 7.2.11. *Stable ∞-category that admits bounded t-structure is idempotent complete.*

Proof. We do induction on $b - a$ to prove that each idempotent in $\mathcal{D}_{[a,b]}$ have splitting in $\mathcal{D}$ and by using shifting, we only need to prove that for $\mathcal{D}_{[0,n]}$ and do inducion on n. The case $n = 0$ is equivalent to the idempotent completeness of $\mathcal{D}^{\heartsuit}$ since it is abelian. Now assume that our claim holds for n and we are going to prove the case $n + 1$. Using the notation of [HTT Section 4.4.5], given any idempotent $\mathrm{Idem} \to \mathcal{D}_{[0,n+1]}$ corresponding to morphism $e : X \to X$, we want to extend it to Idem^{+}. The functoriality of truncations leads to a diagram Idem $\to \mathcal{D}^{\Delta^1}$ corresponding to the right square of the following diagram of fiber sequences:

$$\begin{array}{ccccc} X & \longrightarrow & \tau_{\leq n}X & \longrightarrow & \tau_{>n}X[1] \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \tau_{\leq n}e} & & \downarrow{\scriptstyle \tau_{>n}e[1]} \\ X & \longrightarrow & \tau_{\leq n}X & \longrightarrow & \tau_{>n}X[1] \end{array}$$

We can extend this Idem $\to \mathcal{D}^{\Delta^1}$ to Idem$^+$ using the fact that this extension is equivalent to certain colimits that exists pointwisely, since such extensions always exist for objects in $\mathcal{D}_{[0,n]}$ and $\mathcal{D}_{[n+1,n+1]} \simeq \Sigma^{n+1}\mathcal{D}_{[0,0]}$ by inductive assumption. Take the fiber of the resulting extension Idem$^+ \to \mathcal{D}^{\Delta^1}$ and we get what we want. □

Remark 7.2.12. *A stable ∞-category is idempotent complete if and only if its homotopy category is idempotent complete by [HA Lemma 1.2.4.6].*

Corollary 7.2.13. *The bounded derived category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ for any small abelian category $\mathcal{A}$ is idempotent complete.*

8 Properties of Derived ∞-Category

8.1 Projective Dimension

Definition 8.1.1. *Given an abelian category $\mathcal{A}$ and objects $X, Y \in \mathcal{A}$, the n^{th}* Ext*-group is defined as:*

$$\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \simeq \pi_0 \operatorname{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^n Y)$$

Remark 8.1.2. *These* Ext*-groups form a bi-variant cohomology theory of $\mathcal{A}$. Notice that, by Lemma 7.2.4, the negative* Ext*-groups all vanish and the 0^{th} group is just* Hom*-group of $\mathcal{A}$.*

Definition 8.1.3. *Let $\mathcal{A}$ be an abelian category. Given an object $X \in \mathcal{A}$, the projective dimension of X is the smallest integer n such that for any $p > n$, we have $\mathrm{Ext}^p_{\mathcal{A}}(X, -) \simeq 0$, or if such integer does not exist, the projective dimension of X is ∞, which we will denote as $\operatorname{proj.dim} X = n$ or ∞.*

Dually, using the vanishing $\mathrm{Ext}^p_{\mathcal{A}}(-, X) \simeq 0$ for $p > n$ or not, we can define the injective dimension of X, writed as $\operatorname{inj.dim} X = n$ or ∞.

Lemma 8.1.4. *Let $\mathcal{A}$ be an abelian category. An object $X \in \mathcal{A}$ has projective dimension $\leq n$ if and only if $\mathrm{Ext}^{n+1}_{\mathcal{A}}(X, -)$ is left exact. Dually, X has injective dimension $\leq n$ if and only if $\mathrm{Ext}^{n+1}_{\mathcal{A}}(-, X) \simeq 0$ is left exact.*

Proof. By using opposite category, we only need to show this for injective dimension. Let $h_X : \mathbf{D}^{\mathrm{b}}(\mathcal{A})^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ denote the $\mathcal{S}\mathrm{p}$-enriched Yoneda embedding: $X \mapsto \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, A)$, and its restriction to $\mathcal{A}$ as h^0_X, which is a δ-functor. Notice that, by our assumption of left exactness and long exact sequences of π_n, the termwise truncation $\tau_{\geq -n}\, h^0_X$ and $\tau_{\leq -(n+1)}\, h^0_X$ are also δ-functors. So by universal property of derived category, the cofiber sequence $\tau_{\geq -n}\, h^0_X \to h^0_X \to \tau_{\leq -(n+1)}\, h^0_X$ extends to $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$, and we have cofiber sequence of exact functors $\mathbf{D}^{\mathrm{b}}(\mathcal{A})^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$, which we write as:

$$h^+_X \to h_X \to h^-_X$$

However, by definition, we know that $\pi_0 h^-_X(X) \simeq 0$, so Yoneda lemma implies that the morphism $h_X \to h^-_X$ is zero. It follows that the cofiber sequence $\Sigma^{-1} h^-_X \to h^+_X \to h_X$ splits and hence we have $h^+_X \simeq \Sigma^{-1} h^-_X \oplus h_X$. Still by definition, $\pi_{-n} h^+_X(X) \simeq 0$ for $n \geq p$ and therefore as a direct summand, the same holds for h_X. □

Definition 8.1.5. *The global dimension of an abelian category $\mathcal{A}$ is the smallest integer such that for any $n > p$ and objects $X, Y \in \mathcal{A}$ we have $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \simeq 0$, or if such integer does not exist, the global dimension of $\mathcal{A}$ is ∞, which we will denote as $\operatorname{gl.dim} X = n$ or ∞.*

Corollary 8.1.6. *Let $\mathcal{A}$ be an abelian category. Then $\operatorname{gl.dim} \mathcal{A} \leq n$ if and only if $\mathrm{Ext}^{n+1}_{\mathcal{A}}(X, Y) \simeq 0$ for any $X, Y \in \mathcal{A}$.*

The argument in the proof of Lemma 5.2.10 can be used here to give a dual of Theorem 7.2.4, if $\mathcal{A}$ has finite global dimension.

Theorem 8.1.7. *Let $\mathcal{A}$ be an abelian category with global dimension $\leq p$. Given any objects $X \in \mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ and $Y \in \mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$, for $n > p$ we have:*

$$\pi_0 \operatorname{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma^n Y) \simeq 0$$

Given $X, Y \in \mathcal{A}$, Let $\mathcal{E}(X, Y)$ denote the 1-category with objects the short exact sequence of the form:

$$0 \to Y \to E \to X \to 0$$

And morphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & \downarrow & & {\scriptstyle id}\downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & E' & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

By 5-lemma, this is actually an 1-groupoid.

Theorem 8.1.8. *We have equivalence:*

$$\operatorname{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, \Sigma Y) \simeq \mathrm{N}(\mathcal{E}(X, Y))$$

In particular, we have $\operatorname{Ext}^1_{\mathcal{A}}(X, Y) \simeq \pi_0 \mathcal{E}(X, Y)$.

Proof. ... □

The notion of projective dimension is intimately related to the concept of projective objects in an abelian category.

Definition 8.1.9. *Let $\mathcal{A}$ be an abelian category, an object $X \in \mathcal{A}$ is projective, if the functor $\operatorname{Hom}_{\mathcal{A}}(X, -)$ is exact.*

Theorem 8.1.10. *Let $\mathcal{A}$ be an abelian category. An object X is projective if and only if $\operatorname{proj.dim} X = 0$.*

Proof. Given any short exact sequence $0 \to A' \to A \to A'' \to 0$, we have long exact sequence:

$$0 \to \operatorname{Hom}_{\mathcal{A}}(X, A') \to \operatorname{Hom}_{\mathcal{A}}(X, A) \to \operatorname{Hom}_{\mathcal{A}}(X, A'') \to \operatorname{Ext}^1_{\mathcal{A}}(X, A') \to \dots$$

Therefore, $\operatorname{Hom}_{\mathcal{A}}(X, -)$ is exact if and only if $\operatorname{Ext}^1_{\mathcal{A}}(X, -)$ is left exact. □

We have the following well-known characterization of zero global-dimensional abelian group.

Definition 8.1.11. *An abelian category $\mathcal{A}$ is split, if all short exact sequences in $\mathcal{A}$ split.*

Theorem 8.1.12. *An abelian category $\mathcal{A}$ has global dimension 0 if and only if it is split.*

Proof. An abelian category $\mathcal{A}$ is split if and only if all of its objects are projective, or equivalently, have projective dimension 0. □

The following lemma will be used later.

Lemma 8.1.13. *Let $\mathcal{A}$ be an abelian category, and $P \in \mathcal{A}$ a projective object. For any object $X \in \mathbf{K}^{\mathrm{b}}(\mathcal{A})$, we have equivalence:*

$$\pi_0 \operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(P, X) \simeq \pi_0 \operatorname{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(P, X) \simeq \operatorname{Hom}_{\mathcal{A}}(P, \mathrm{H}_0(X))$$

Proof. Let $\mathcal{D}$ denote the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ comprised of objects X such that we have the following equivalences for any n:

$$\pi_0 \operatorname{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(P, \Sigma^{-n} X) \simeq \operatorname{Hom}_{\mathcal{A}}(P, \mathrm{H}_n(X))$$

The long exact sequences and 5-lemma imply that $\mathcal{D}$ is stable and the vanishing of Ext for P implies that $\mathcal{A} \subseteq \mathcal{D}$. Therefore $\mathcal{D} \simeq \mathbf{D}^{\mathrm{b}}(\mathcal{A})$. This shows the second equivalence, and the first would follows from a similar argument. □

Given an abelian category $\mathcal{A}$, the additive subcategory of projective objects will be denoted as $\mathcal{A}_{\mathrm{proj}}$. We can deduce the following corollary immediately.

Corollary 8.1.14. *Let $\mathcal{A}$ be an abelian category. Given an object $X \in \mathbf{K}^{\mathrm{b}}(\mathcal{A}_{proj})$, we have the following equivalence for any object $Y \in \mathbf{K}^{\mathrm{b}}(\mathcal{A})$:*

$$\operatorname{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X, Y) \simeq \operatorname{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, Y)$$

8.2 Projective Resolutions

We say $\mathcal{A}$ admits enough projectives, if for any object $X \in \mathcal{A}$, there exists an epimorphism $P \to X$ such that P is projective.

Theorem 8.2.1. *Let $\mathcal{A}$ be an abelian category that admits enough projectives. The exact functor $j : \mathbf{K}^{\mathrm{b}}(\mathcal{A}_{\mathrm{proj}}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ induced by universal property is fully faithful. Moreover, j is an equivalence of categories if and only if all objects in $\mathcal{A}$ have finite projective dimension.*

Proof. The fully-faithfulness of j is implied by Remark 5.2.8. First, suppose that all objects in $\mathcal{A}$ have finite projective dimension. To show the last claim, let $\mathcal{A}_n$ denote the subcategory of $\mathcal{A}$ comprised of objects of projective dimension $\leq n$. We will use induction to show that $\mathcal{A}_n \subseteq \mathbf{K}^{\mathrm{b}}(\mathcal{A}_{\mathrm{proj}})$. Notice that $\mathcal{A}_0 \simeq \mathcal{A}_{\mathrm{proj}}$, so the case $n = 0$ is obvious. Assume that $\mathcal{A}_n \subseteq \mathbf{K}^{\mathrm{b}}(\mathcal{A}_{\mathrm{proj}})$, given any object $X \in \mathcal{A}_{n+1}$, we take an epimorphism from a projective $P \to X$ and take its kernel to form a short exact sequence:

$$0 \to K \to P \to X \to 0$$

Using long exact sequence of Ext and vanishing of Ext for projectives, we have equivalences for any $p > 0$:

$$\mathrm{Ext}^p_{\mathcal{A}}(K, -) \simeq \mathrm{Ext}^{p+1}_{\mathcal{A}}(X, -)$$

It follows that $K \in \mathcal{A}_n$, hence $X \simeq \mathrm{cofib}(K \to P) \in \mathbf{K}^{\mathrm{b}}(\mathcal{A}_{\mathrm{proj}})$. By our assumption, $\mathcal{A} \subseteq \mathbf{K}^{\mathrm{b}}(\mathcal{A}_{\mathrm{proj}})$ and the theorem follows from that the smallest stable subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ that contains $\mathcal{A}$ is $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ itself.

For the other direction, the full subcategory of objects X that $\mathrm{Ext}^p_{\mathcal{A}}(X, -)$ is non-zero only for finitely many p is stable in $\mathbf{K}^{\mathrm{b}}(\mathcal{A}_{\mathrm{proj}})$ and contains $\mathcal{A}_{\mathrm{proj}}$, so it is just $\mathbf{K}^{\mathrm{b}}(\mathcal{A}_{\mathrm{proj}})$. Therefore any object in $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ satisfies this property, let alone $\mathcal{A}$. ∎

8.3 Recognition of Bounded Derived ∞-Category

In this section, we give some results to decide whether a stable ∞-category with t-structure is equivalent to the bounded derived category of its heart, following the main ideas of the classic [BBD]. We need some special properties that hold in the case of derived category first.

Definition 8.3.1. *Given a stable ∞-category $\mathcal{D}$ and a full subcategory $\mathcal{E} \subseteq \mathcal{D}$, we say that $\mathrm{Ext}^*_{\mathcal{E}}$ is generated by $\mathrm{Ext}^1_{\mathcal{E}}$ if any morphism $X \to Y[n+1]$ $(n > 0)$ with $X, Y \in \mathcal{E}$ can be factorized as following with $Z_i \in \mathcal{E}$:*

$$X \to Z_1[1] \to Z_2[2] \to \cdots \to Z_n[n] \to Y[n+1]$$

Remark 8.3.2. *To prove that $\mathrm{Ext}^*_{\mathcal{E}}$ is generated by $\mathrm{Ext}^1_{\mathcal{E}}$, it is enough to show that any morphism $X \to Y[n+1]$ $(n > 0)$ with $X, Y \in \mathcal{E}$ can be factorized as with $Z \in \mathcal{E}$:*

$$X \to Z[n] \to Y[n+1]$$

Or dually:

$$X \to Z[1] \to Y[n+1]$$

Lemma 8.3.3. *Given a small abelian category $\mathcal{A}$, and an object $X \in \mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$. There exists a cofiber sequence with $A \in \mathcal{A}$ and $X' \in \mathbf{D}^{\mathrm{b}}_{\geq 1}(\mathcal{A})$:*

$$A \to X \to X'$$

Proof. Using Corollary 5.3.8. □

Theorem 8.3.4. *We have that $\mathrm{Ext}^*_{\mathcal{A}}$ is generated by $\mathrm{Ext}^1_{\mathcal{A}}$ in the bounded derived category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ for any small abelian category $\mathcal{A}$.*

Proof. Given morphism $X \to Y[n]$ $(n > 1)$ with $X, Y \in \mathcal{A}$ in $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$, we take the fiber as F:

$$F \to X \to Y[n]$$

By definition $F \in \mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and we can use the previous lemma to find object $A \in \mathcal{A}$ and morphism $A \to F$ of which cofiber lies in $\mathbf{D}^{\mathrm{b}}_{\geq 1}(\mathcal{A})$. The long exact sequence of homology groups shows that the composition $A \to F \to X$ is epimorphism in $\mathcal{A}$ hence its cofiber is of form $A'[1]$ with $A' \in \mathcal{A}$. The composition $A \to X \to Y[n]$ vanishes so we have factorization $X \to A'[1] \to Y[n]$. □

Theorem 8.3.5. *Two factorization of $\mathrm{Ext}^n_{\mathcal{A}}$ by $\mathrm{Ext}^1_{\mathcal{A}}$ in the bounded derived category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ for any small abelian category $\mathcal{A}$.*

Lemma 8.3.6. *Given a small abelian category $\mathcal{A}$, a stable ∞-category $\mathcal{D}$ with t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$, and an exact functor $f : \mathcal{A} \to \mathcal{D}^{\heartsuit}$, by universal property, we get an exact functor $F : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$, and the following holds:*

1. *If f is fully faithful, the functor F induces injection:*

$$\mathrm{Ext}^1_{\mathcal{A}}(X,Y) \to \pi_{-1}\,\mathrm{Map}_{\mathcal{D}}(F(X),F(Y))$$

2. *If f is fully faithful, and the essential image $f(\mathcal{A})$ is closed under extension, the functor F induces isomorphism:*

$$\mathrm{Ext}^1_{\mathcal{A}}(X,Y) \simeq \pi_{-1}\,\mathrm{Map}_{\mathcal{D}}(F(X),F(Y))$$

3. *If for any $0 \le i \le n$, the functor F induces isomorphisms:*

$$\mathrm{Ext}^i_{\mathcal{A}}(X,Y) \simeq \pi_{-i}\,\mathrm{Map}_{\mathcal{D}}(F(X),F(Y))$$

   Then we have injection:

$$\mathrm{Ext}^{n+1}_{\mathcal{A}}(X,Y) \to \pi_{-(n+1)}\,\mathrm{Map}_{\mathcal{D}}(F(X),F(Y))$$

Theorem 8.3.7. *Given a small stable ∞-category $\mathcal{D}$ with t-structure $(\mathcal{D}_{\ge 0},\mathcal{D}_{\le 0})$, the canonical functor $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\heartsuit}) \to \mathcal{D}$ is equivalence if and only if the following holds:*

1. *The t-structure is bounded, or equivalently, the smallest stable subcategory that contains $\mathcal{D}^{\heartsuit}$ is $\mathcal{D}$ itself;*
2. *We have that $\mathrm{Ext}^*_{\mathcal{D}^{\heartsuit}}$ is generated by $\mathrm{Ext}^1_{\mathcal{D}^{\heartsuit}}$.*

Remark 8.3.8. *The condition (2) is equivalent to that the following induced maps are all surjective:*

$$\pi_0\,\mathrm{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\heartsuit})}(X,Y[n]) \to \pi_0\,\mathrm{Map}_{\mathcal{D}}(X,Y[n])$$

Notice that, if the t-structure is not bounded, we are still able to deduce the fully-faithfulness from (2).

Remark 8.3.9. *Once we have the functor $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\heartsuit}) \to \mathcal{D}$, these conditions can be formulated and checked totally inside the homotopy category $h\mathcal{D}$ using its canonical triangulated structure, without any referring to the ∞-category $\mathcal{D}$. However, to construct that functor, we need more structures than merely the homotopy category.*

Another criterion is:

Theorem 8.3.10. *Let $\mathcal{D}$ be a small stable ∞-category with t-structure. The canonical functor $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\heartsuit}) \to \mathcal{D}$ is equivalence if and only if it is essentially surjective.*

Proof. We only need to prove that any morphism $X \to Y[n]$ $(n > 1)$ for $X, Y \in \mathcal{D}^{\heartsuit}$ can be lifted to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\heartsuit})$. Given $X \to Y[n]$ in $\mathcal{D}$, by essential surjectivity, its fiber F is equivalent to some object in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\heartsuit})$. The morphism $F \to X$ can be lifted since it is defined by truncations in the t-structure. □

We can deduce some immediate corollaries that may have some interest.

Corollary 8.3.11. *Given a small stable ∞-category $\mathcal{D}$ with bounded t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ such that $\mathrm{Ext}^i_{\mathcal{D}^\heartsuit}(X, Y) \simeq 0$ for $n > 1$, then the canonical functor $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^\heartsuit) \to \mathcal{D}$ is equivalence.*

Corollary 8.3.12. *Given a pullback square of small abelian categories with exact functors between them:*

$$\begin{array}{ccc} \mathcal{A}' \times_{\mathcal{A}} \mathcal{A}'' & \longrightarrow & \mathcal{A}' \\ \downarrow & & \downarrow \\ \mathcal{A}'' & \longrightarrow & \mathcal{A} \end{array}$$

If $\mathrm{gl.dim}\,\mathcal{A} = 0$ *and* $\mathrm{gl.dim}\,\mathcal{A}', \mathcal{A}'' \leq 1$*, the induced square of bounded derived ∞-categories is pullback of stable ∞-categories:*

$$\begin{array}{ccc} \mathbf{D}^{\mathrm{b}}(\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}'') & \longrightarrow & \mathbf{D}^{\mathrm{b}}(\mathcal{A}') \\ \downarrow & & \downarrow \\ \mathbf{D}^{\mathrm{b}}(\mathcal{A}'') & \longrightarrow & \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \end{array}$$

8.4 Uniqueness of Enhancement

The universal property of bounded derived category has an interesting consequence. An ∞-categrical enhancement of a triangulated category $\mathcal{T}$ is a stable ∞-category $\mathcal{D}$ with a triangulated equivalence $h\mathcal{D} \simeq \mathcal{T}$. In general, many different enhancements arise possibly. However, it has to be unique in the case of derived categories. Before proving this fact, we establish in some sense a weak universal property for $h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A})$. Especially, it shows that if a triangulated category $\mathcal{D}$ with t-structure admits at least one ∞-categorical enhancement, any exact functor $\mathcal{A} \to \mathcal{D}^{\heartsuit}$ can be extended to triangulated functor from $h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A})$.

Lemma 8.4.1. *Given a stable ∞ category $\mathcal{D}$, a small abelian category $\mathcal{A}$ and a functor $f : \mathcal{A} \to h\mathcal{D}$ satisfying the following properties:*

i. For any $X, Y \in \mathcal{A}$, we have vanishing for $n > 0$:

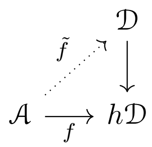
$$\mathrm{Hom}_{h\mathcal{D}}(f(X), f(Y)[-n]) \simeq 0$$

ii. For any short exact sequence $0 \to X \to Y \to Z \to 0$, its image under f can be extended to a distinguished triangle:

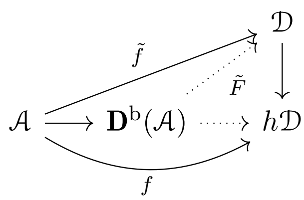
$$f(X) \to f(Y) \to f(Z) \to f(X)[1]$$

Then we have essentially unique 2-step extension:

1. The functor f has an essentially unique lifting to $\mathcal{D}$, and any such lifting is automatically a δ-functor (cf. Definition 7.0.1):



2. The resulting diagram in (1) has essentially unique extension to the following diagram with $\tilde{F}$ an exact functor:



Proof. (1) Let $f(\mathcal{A}) \subseteq h\mathcal{D}$ be the essential image of $\mathcal{A}$ under f, and $\widetilde{f(\mathcal{A})} \subseteq \mathcal{D}$ the full subcategory consists of objects that contained in $f(\mathcal{A})$ in homotopy category. Then the lifting problem is equivalent to:

$$\begin{array}{ccc} & & \widetilde{f(\mathcal{A})} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow t \\ \mathcal{A} & \xrightarrow[f]{} & f(\mathcal{A}) \end{array}$$

Notice that, by property (i), $\widetilde{f(\mathcal{A})}$ is 1-category, and therefore t is equivalence of categories. It follows that the space of this lifting problem is contractible.

Now we show that any such extension $\tilde{f}$ is automatically a δ-functor. Actually, property (ii) shows that for any short exact sequence, there exists a null-homotopy in $\mathcal{D}$ to fill-in the following square that can make it a cofiber sequence:

$$\begin{array}{ccc} \tilde{f}(X) & \longrightarrow & \tilde{f}(Y) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{f}(Z) \end{array}$$

However, any null-homotopy meets our requirement, including the one given by the image of this short exact sequence under $\tilde{f}$. The reason is, the space of such null-homotopies is the path space between null-homotopic morphisms in $\mathrm{Map}_{\mathcal{D}}(\tilde{f}(X), \tilde{f}(Z))$, which is contractible by (i).

(2) Since $\tilde{f}$ is a δ-functor, the universal property of bounded derived category provides us an exact functor $\tilde{F} : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$. The diagram presented in (2) is indexed by Δ^3. The sub-diagram indexed by $\partial\Delta^3$ is commutative by definition, and it extends essentially uniquely to the full diagram since $h\mathcal{D}$ is 1-category. $\square$

The uniqueness of enhancement now follows easily.

Theorem 8.4.2. *The triangulated category $h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ for small abelian category $\mathcal{A}$ has essentially unique ∞-categrical enhancement $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$.*

Proof. Given a stable ∞-category $\mathcal{D}$ and a triangulated equivalence $h\mathcal{D} \simeq h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A})$, we apply the previous lemma with f being the canonical inclusion $\mathcal{A} \to h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A}) \simeq h\mathcal{D}$. Notice that, the canonical t-structure of $h\mathcal{D} \simeq h\,\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ gives $\mathcal{D}$ a corresponding t-structure and the functor $\tilde{f}$ induces equivalence $\mathcal{D}^{\heartsuit} \simeq \mathcal{A}$. The resulting exact functor $\tilde{F} : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ is equivalent to the one given in Theorem 8.3.7 and by the same theorem, $\tilde{F}$ is an equivalence of categories (cf. Remark 8.3.9). $\square$

8.5 Derived Category of Finite Diagrams

We will write $\mathcal{C}^K$ for $\mathrm{Fun}(K, \mathcal{C})$ in this section.

Definition 8.5.1. *A finite* 1*-category I is an* 1*-category with finitely many isomorphic class of objects and finite* Hom*-sets.*

The canonical t-structure of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ induces a bounded t-structure on $\mathbf{D}^{\mathrm{b}}(\mathcal{A})^I$ with heart equivalent to $\mathcal{A}^I$. Therefore we can use universal property to construct exact and t-exact functor $\mathbf{D}^{\mathrm{b}}(\mathcal{A}^I) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})^I$ which induces equivalence of hearts.

Theorem 8.5.2. *Let $\mathcal{A}$ be a small abelian category $\mathcal{A}$ and I a finite* 1*-category. We have canonical equivalence:*

$$\mathbf{D}^{\mathrm{b}}(\mathcal{A}^I) \simeq \mathbf{D}^{\mathrm{b}}(\mathcal{A})^I$$

The proof will be given in the end of this section. We prove a lemma first. Let $\mathcal{C}$ be an ∞-category that admits finite limits, and I a finite 1-category. For any object $i \in I$ and object $X \in \mathcal{C}$, the right Kan extension of X (seen as diagram $\Delta^0 \to \mathcal{C}$) along $i : \Delta^0 \to I$ exists since its value on any object $j \in I$ is finite product of X indexed by $\mathrm{Hom}_I(j, i)$. Hence we have a pair of adjoint functors:

$$\mathcal{C}^I \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{C}$$

Here i^* means restriction to value at i, namely $i^*F = F(i)$, and i_* the right Kan extension along $i : \Delta^0 \to I$. Moreover, if $\mathcal{C}$ is an abelian category, i_* is exact since its construction only uses finite products which is exact. The universal property of derived categories induces a pair of adjoint functors, that both are exact and t-exact, for small abelian category $\mathcal{A}$:

$$\mathbf{D}^{\mathrm{b}}(\mathcal{A}^I) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}(\mathcal{A})$$

It follows that, for any diagram $F : I \to \mathcal{A}$ and object $A \in \mathcal{A}$, we have equivalence of mapping spectra (we use $\mathcal{M}\mathrm{ap}$ for mapping spectrum and Map for mapping space):

$$\mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A}^I)}(F, i_*A) \simeq \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(F(i), A)$$

Notice that, we can also apply such right Kan extensions to $\mathbf{D}^{\mathrm{b}}(\mathcal{A})^I$. Also by the same reason that it is defined by finite products, the right Kan extension functor i_* is t-exact:

$$\mathbf{D}^{\mathrm{b}}(\mathcal{A})^I \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}(\mathcal{A})$$

Similarly, we have equivalence of mapping spectra:

$$\mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})^I}(F, i_*A) \simeq \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(F(i), A)$$

The above discussion is summarized as:

Lemma 8.5.3. *We have equivalence of mapping spectrum for any diagram $F : I \to \mathcal{A}$ and object $A \in \mathcal{A}$:*

$$\mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A}^I)}(F, i_*A) \simeq \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})^I}(F, i_*A)$$

Now let $GF \simeq \bigoplus_{i\in I} i_*i^*F$ (the direct sum is taken among isomorphic classes of objects in I). Since I is finite, this is a finite direct sum. The construction G is a functor and we have natural transformation $id_{\mathcal{A}} \to G$.

Lemma 8.5.4. *The map $F \to GF$ is monomorphism for any diagram F.*

Proof. For each object $j \in I$ the value at i of this map factorized as:

$$F(j) \to j_*F(j) \hookrightarrow \bigoplus_{i\in I} i_*i^*F$$

And the map $F(j) \to j_*F(j)$ is equivalent to:

$$F(j) \hookrightarrow \bigoplus_{f\in \mathrm{Hom}_I(j,j)} F(j) \simeq j_*F(j)$$

□

Proof of Theorem 8.5.2. The previous Lemma 8.5.4 shows that for any diagram $F : I \to \mathcal{A}$, we have short exact sequence $0 \to F \to GF \to F' \to 0$ and it induces natural transformation between fiber sequences of mapping spectra for any other diagram $T : I \to \mathcal{A}$, in which the middle vertical map is equivalence by Lemma 8.5.3:

$$\begin{array}{ccccc}
\mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A}^I)}(T, F) & \longrightarrow & \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A}^I)}(T, GF) & \longrightarrow & \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A}^I)}(T, F') \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})^I}(T, F) & \longrightarrow & \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})^I}(T, GF) & \longrightarrow & \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})^I}(T, F')
\end{array}$$

Now we can do induction. Theorem 8.3.6 shows that the left and right vertical map induces isomorphisms on π_i for $i \leq 0$, monomorphisms on π_{-1} and Lemma 8.5.3 implies that the middle vertical map is equivalence. Hence by five lemma, the left vertical map induces isomorphism on π_{-1}. However, this claim holds for arbitrary choice of F. Therefore the right vertical map also induces isomorphisms on π_{-1}. It is implied by (3) of Theorem 8.3.6 that the left and right vertical maps induce monomorphisms on π_{-2}. Proceeding like this, we conclude that the left and right vertical maps induces isomorphisms on homotopy groups and hence are equivalences. It follows that, the comparison functor $\mathbf{D}^{\mathrm{b}}(\mathcal{A}^I) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})^I$ is fully faithful on the heart $\mathcal{A}^I$ and hence fully faithful on the entire derived category. The boundedness of the induced t-structure on $\mathbf{D}^{\mathrm{b}}(\mathcal{A})^I$ implies that the comparison functor is also essentially surjective. □

Example 8.5.5. *Let G be a finite group, then the groupoid BG is a finite 1-category. We have equivalence for any commutative ring R:*

$$\mathbf{D}^{\mathrm{b}}(R)^{BG} \simeq \mathbf{D}^{\mathrm{b}}(R\,\mathbf{Mod}^{BG}) \simeq \mathbf{D}^{\mathrm{b}}(R[G])$$

8.6 Filtered Derived Category

Definition 8.6.1. *Let $\mathcal{A}$ be a small abelian category. The n-filtered category of $\mathcal{A}$, denoted by* $\mathrm{Fil}_n(\mathcal{A})$ *is the full subcategory of* $\mathcal{A}^{\Delta^n}$ *consists of n-consecutive monomorphisms:*

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n$$

The filtered derived category

Remark 8.6.2. *Notice that* $\mathrm{Fil}_n(\mathcal{A})$ *is not abelian unless $\mathcal{A}$ is trivial. However, it is an exact category.*

Definition 8.6.3. *Let $\mathcal{A}$ be a small stable ∞-category. The n-filtered category of $\mathcal{A}$, denoted by* $\mathrm{Fil}_n(\mathcal{A})$ *is the full subcategory of* $\mathcal{A}^{\Delta^n}$ *consists of n-consecutive monomorphisms:*

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n$$

8.7 Derived Auslander's Formula

In this section, we give new proof of the result of [Krause]. We claim no originality in the results in this section.

9 Derived Functor

Given an abelian category $\mathcal{A}$ and a stable ∞-category $\mathcal{D}$ with t-structure, sny additive functor $F : \mathcal{A} \to \mathcal{D}^{\heartsuit}$ can be extended in an essentially unique way to an exact functor by universal property (by abuse of notation we still write this as F):

$$F : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$$

If f is not exact, we cannot use the universal property of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ to further extend it to the derived category. However, if $\mathcal{D}$ admits small colimits or limits, we can use left or right Kan extensions to obtain exact functor which are called right or left derived functor:

$$RF, LF : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$$

By convention, we will use notations $R^n F(X) \simeq \pi_{-n} RF(X)$ and $L_n F(X) \simeq \pi_n LF(X)$.

Theorem 9.0.1. *Let $F : \mathcal{A} \to \mathcal{D}^{\heartsuit}$ be an additive functor such that $\mathcal{D}$ admits filtered colimits. The right derived functor $RF : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ exists and the following hold:*

1. *The functor RF is exact;*
2. *If $\mathcal{D}_{\leq 0}$ is closed under filtered colimits, RF is left t-exact;*
3. *If $\mathcal{D}_{\leq 0}$ is closed under filtered colimits and F is left exact, we have canonical equivalence for any $X \in \mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$:*

$$\pi_0 RF(X) \simeq F(\mathrm{H}_0(X))$$

Proof. Exactness is implied by Theorem 4.5.1. Given any object $X \in \mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$, we can find object $\bar{X} \in \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ and the value of $LR(X)$ can be computes as a filtered colimit (cf. Theorem 1.1.2 (2), proof of Theorem 7.2.2 (2) and Theorem 1.3.1):

$$RF(X) \simeq \mathrm{colim}_{\bar{X} \to Y \in (\mathcal{Q}_{\leq 0})_{\bar{X}/}} F(Y)$$

In particular, $RF(X)$ can be represented as filtered colimits of objects $F(Y)$ with $Y \in \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$. Since F maps $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ into $\mathcal{D}_{\leq 0}$ by definition and $\mathcal{D}_{\leq 0}$ is closed under colimits, we have $RF(X) \in \mathcal{D}_{\leq 0}$, which shows the left t-exactness of RF.

Now assume that F is left exact. First we show that for any $X \in \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$, the following equivalence induced by $\mathrm{H}_0(X) \to X$ holds:

$$\pi_0 F(X) \simeq F(\mathrm{H}_0(X))$$

The proof is using induction. Let $\mathcal{C}$ denote the full subcategory of $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ consists of objects X for which the above equivalence holds. We have $\mathcal{A} \subseteq \mathcal{C}$ by definition. And since $\pi_0 F(-)$ and $F(\mathrm{H}_0(-))$ are both left exact functors (seen as functors $\mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A}) \to \mathcal{D}^{\heartsuit}$), five lemma shows that $\mathcal{C}$ is closed under finite limits. It follows that $\mathcal{C} \simeq \mathbf{K}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$.

Since $\mathcal{D}_{\leq 0}$ is closed under filtered colimits, π_0 commutes with filtered colimits. The previous discussion shows that:

$$\pi_0 RF(X) \simeq \mathrm{colim}_{\bar{X} \to Y \in (\mathcal{Q}_{\leq 0})_{\bar{X}/}} \pi_0 F(Y)$$

Now since morphisms in $\mathcal{Q}_{\leq 0}$ are quasi-isomorphisms we have $\pi_0 F(Y) \simeq \pi_0 F(\bar{X}) \simeq F(\mathrm{H}_0(X))$ and the morphisms in diagram $Y \mapsto \pi_0 F(Y)$ are all isomorphisms. The contractibility of $(\mathcal{Q}_{\leq 0})_{\bar{X}/}$ implies that the right-hand-side colimit is just $F(\mathrm{H}_0(X))$. □

Theorem 9.0.2. *Let $\mathcal{A}$ be a small abelian category and $\mathcal{D}$ a stable ∞-category that admits filtered limits and with t-structure that $\mathcal{D}_{\leq 0}$ is closed under filtered colimits. Then taking right derived functor gives fully faithful functor:*

$$R(-) : \mathrm{Fun}^{lex}(\mathcal{A}, \mathcal{D}^{\heartsuit}) \hookrightarrow \mathrm{Fun}^{t-lex}(\mathbf{D}^{\mathrm{b}}(\mathcal{A}), \mathcal{D})$$

Proof. Actually a more general proposition holds. Let $F, G : \mathcal{A} \to \mathcal{D}^{\heartsuit}$ be additive functors and we only require G is left exact. We write $q : \mathbf{K}^{\mathrm{b}}(\mathcal{A}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ for the canonical functor and $q_!, q^*$ the left Kan extension and restriction along q (hence $q_! F \simeq RF$). Notice that they form pair of adjoints and so we have equivalence:

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}),\mathcal{D})}(F, q^* q_! G) \simeq \mathrm{Map}_{\mathrm{Fun}^{\mathrm{ex}}(\mathbf{D}^{\mathrm{b}}(\mathcal{A}),\mathcal{D})}(q_! F, q_! G)$$

By definition, F has its values in $\mathcal{D}^{\heartsuit} \subseteq \mathcal{D}_{\geq 0}$ and by (2) of Theorem 9.0.1, $q^* q_! G$ has its values in $\mathcal{D}_{\leq 0}$. Therefore we have equivalence:

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}),\mathcal{D})}(F, q^* q_! G) \simeq \mathrm{Map}_{\mathrm{Fun}^{\mathrm{ex}}(\mathbf{K}^{\mathrm{b}}(\mathcal{A}),\mathcal{D})}(F, \tau_{\leq 0}(q^* q_! G))$$

The claim (2) of Theorem 9.0.1 implies $G \simeq \tau_{\leq 0}(q^* q_! G)$ and hence after restricting back to $\mathcal{A}$, the universal property of $\mathbf{K}^{\mathrm{b}}$ justifies our proposition. □

If $\mathcal{A}$ admits enough injectives, we have a criterion to decide whether a right t-exact functor $G : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ is equivalent to $R(\pi_0 G|_{\mathcal{A}})$.

Theorem 9.0.3. *Given right t-exact functor $G : \mathbf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}$ such that $\mathcal{A}$ admits enough injectives, $\mathcal{D}$ admits filtered colimits and $\mathcal{D}_{\leq 0}$ is closed under filtered colimits. then $G \simeq R(\pi_0 G|_{\mathcal{A}})$ if and only if G sends injective objects of $\mathcal{A}$ into the heart of $\mathcal{D}$.*

Proof. First we prove that, for any left exact functor $F : \mathcal{A} \to \mathcal{D}^{\heartsuit}$, the functor RF maps injective object I into $\mathcal{D}^{\heartsuit}$. By definition, left Kan extension is computed by a colimit indexed by $\mathbf{K}^{\mathrm{b}}(\mathcal{A})_{/q(I)}$. However, the dual of Corollary 8.1.14 shows that $\mathbf{K}^{\mathrm{b}}(\mathcal{A})_{/q(I)}$ admits a terminal object, which is the identity $id_{q(I)}$. Hence we have $RF(I) \simeq F(I) \in \mathcal{D}^{\heartsuit}$.

For the converse, we have a canonical morphism $G(X) \to R(\pi_0 G|_{\mathcal{A}})(X)$ induced by the universal property of Kan extension. For the sake of simplicity, we use G_0 for $\pi_0 G|_{\mathcal{A}}$ from now on. Since $\mathcal{A}$ generates $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$, we only need to show that $G(X) \to RG_0(X)$ is equivalence for $X \in \mathcal{A}$. The assumption that $\mathcal{A}$ admits enough injectives allows us to find an exact sequence $0 \to X \to I \to X' \to 0$ with I injective. We have natural transformation between fiber sequences:

$$\begin{array}{ccccc} G(X) & \longrightarrow & G(I) & \longrightarrow & G(X') \\ \downarrow & & \downarrow & & \downarrow \\ RG_0(X) & \longrightarrow & RG_0(I) & \longrightarrow & RG_0(X') \end{array}$$

Notice that for any $X \in \mathcal{A}$, $\pi_{-n}G(X) \simeq \pi_{-n}RG_0(X) \simeq 0$ when $n < 0$ and $\pi_0 G(X) \simeq \pi_0 RG_0(X)$. Also, for any injective I, $\pi_{-n}G(I) \simeq \pi_{-n}RG_0(I) \simeq 0$ unless $n = 0$. Therefore we can use induction, that begin with $n = 0$, using the long exact sequences induced by the above diagram, to show that $\pi_{-n}G(X) \simeq \pi_{-n}RG_0(X)$ for all n. It follows that $G(X) \simeq RG_0(X)$. $\square$

Remark 9.0.4. *Under the assumption of the previous theorem, the essential image of the right derived procedure consists of functors that maps injective objects of $\mathcal{A}$ towards the heart of $\mathcal{D}$.*

We can also define derived multi-functor by using Kan extension of multi-functors (cf. Theorem 4.5.2) with almost verbatim properties and proof.

Theorem 9.0.5. *Let $F : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \to \mathcal{D}^{\heartsuit}$ be an additive multi-functor such that $\mathcal{D}$ admits filtered colimits. The right derived multi-functor $RF : \mathbf{D}^{\mathrm{b}}(\mathcal{A}_1) \times \mathbf{D}^{\mathrm{b}}(\mathcal{A}_2) \times \cdots \times \mathbf{D}^{\mathrm{b}}(\mathcal{A}_n) \to \mathcal{D}$ exists and the following hold:*

1. *The functor RF is exact;*
2. *If $\mathcal{D}_{\leq 0}$ is closed under filtered colimits, RF is left t-exact;*
3. *If $\mathcal{D}_{\leq 0}$ is closed under filtered colimits and F is left exact, we have canonical equivalence for any series of objects $X_i \in \mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A}_i)$:*

$$\pi_0 RF(X_1, X_2, \ldots, X_n) \simeq F(\mathrm{H}_0(X_1), \mathrm{H}_0(X_2), \ldots, \mathrm{H}_0(X_n))$$

Propositions similar to Theorem 9.0.2 and Theorem 9.0.3 also hold.

9.1 Derived Mapping Spectrum

As an application, we will re-prove a classical proposition that the mapping space in derived category is the right derived mapping complexes, without referring to any projective or injective resolutions. Then we use this to show that derived category is enriched over $\mathbb{Z}$-complexes, as is expected from the conventional definition.

Let $\mathcal{A}$ be any small abelian category $\mathcal{A}$. The stable ∞-category of spectra $\mathcal{S}\mathrm{p}$ admits a canonical t-structure with heart equivalent to the category of abelian groups, and $\mathcal{S}\mathrm{p}_{\leq 0}$ is stable under filtered colmits. Therefore the right derived bi-functor of $\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Ab} \simeq \mathcal{S}\mathrm{p}^{\heartsuit}$ exists.

Theorem 9.1.1. *We have canonical equivalence:*

$$R\,\mathrm{Hom}_{\mathcal{A}}(X, Y) \simeq \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, Y)$$

Proof. The equivalence (cf.) $\mathrm{Hom}_{\mathcal{A}}(X, Y) \simeq \tau_{\leq 0}\,\mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}$ for $X, Y \in \mathcal{A}$ induces comparison natural transformation $R\,\mathrm{Hom}_{\mathcal{A}} \to \mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}$ by universal property of Kan extensions. By definition we have:

$$\begin{aligned} R^n\,\mathrm{Hom}_{\mathcal{A}}(X, Y) &\simeq \mathrm{colim}_{X \to X' \in (\mathcal{Q}_{\leq 0})^{\mathrm{op}}_{/X}}\, \mathrm{colim}_{Y \to Y' \in (\mathcal{Q}_{\leq 0})_{Y/}}\, \pi_0\,\mathrm{Map}_{\mathbf{K}^{\mathrm{b}}(\mathcal{A})}(X', Y'[n]) \\ &\simeq \pi_0\,\mathrm{Map}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, Y[n]) \\ &\simeq \pi_{-n}\,\mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}(X, Y) \end{aligned}$$

Therefore the comparison natural transformation induces isomorphisms of homotopy groups and is equivalence. □

The category of abelian groups $\mathbf{Ab}$ is also the heart of the derived category $\mathbf{D}(\mathbb{Z})$ with canonical t-structure. We have functor $i_* : \mathbf{D}(\mathbb{Z}) \to \mathcal{S}\mathrm{p}$ that sends abelian groups A to Eilenberg-MacLane spectrum $\mathrm{H}A$. We can derived the functor $\mathrm{Hom}_{\mathcal{A}}$ with target not $\mathcal{S}\mathrm{p}$ but $\mathbf{D}(\mathbb{Z})$. Since the functor i_* preserves filtered colimits, it also preserves right derived functors. The universal property implies that $\mathcal{M}\mathrm{ap}_{\mathbf{D}^{\mathrm{b}}(\mathcal{A})}$ has an essentially unique lifting:

$$\begin{array}{ccc} & & \mathbf{D}(\mathbb{Z}) \\ & \nearrow & \downarrow{\scriptstyle i_*} \\ \mathbf{D}^{\mathrm{b}}(\mathcal{A})^{\mathrm{op}} \times \mathbf{D}^{\mathrm{b}}(\mathcal{A}) & \longrightarrow & \mathcal{S}\mathrm{p} \end{array}$$

Theorem 9.1.2. *The derived category $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ has essentially unique $\mathbf{D}(\mathbb{Z})$-enrichment structure that makes $\mathcal{A} \to \mathbf{D}^{\mathrm{b}}(\mathcal{A})$ to be $\mathbf{D}(\mathbb{Z})$-enriched functor.*

9.2 Deligne Tensor Product of Small Abelian Categories

The following definition is from [Deligne]:

Definition 9.2.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be small abelian categories. The Deligne tensor product $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is a right-exact bi-functor:*

$$\boxtimes : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_1 \boxtimes \mathcal{A}_2$$

such that it induces equivalence of categories for small abelian category $\mathcal{A}'$:

$$\boxtimes^* : \mathrm{Fun}^{rex}(\mathcal{A}_1 \boxtimes \mathcal{A}_2, \mathcal{A}') \to \mathrm{Fun}^{rex}((\mathcal{A}_1, \mathcal{A}_2), \mathcal{A}')$$

One can say that $\boxtimes$ is the initial right-exact bi-functor.

10 Unbounded Derived ∞-Category

10.1 Unseparated Derived Category

We will use $\mathrm{Fun}^{\mathbf{L}}(\mathcal{C}, \mathcal{C}')$ to denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ comprised of functors that preserve small colimits and $\mathrm{Fun}^{\delta}_{\mathrm{con}}(\mathcal{A}, \mathcal{C})$ the full subcategory of $\mathrm{Fun}^{\delta}(\mathcal{A}, \mathcal{C})$ of functors that preserve filtered colimits.

Definition 10.1.1. *Let $\mathcal{A}$ be an abelian category that admits small colimits. The unseparated derived ∞-category is the universal δ-functor that preserves filtered colimits towards a stable ∞-category that admits small colimits $i : \mathcal{A} \to \check{\mathbf{D}}(\mathcal{A})$, namely it induces equivalence of category for any stable ∞-category $\mathcal{D}$ that admits small colimits:*

$$i^* : \mathrm{Fun}^{\mathbf{L}}(\check{\mathbf{D}}(\mathcal{A}), \mathcal{D}) \simeq \mathrm{Fun}^{\delta}_{\mathrm{con}}(\mathcal{A}, \mathcal{D})$$

Theorem 10.1.2. *Let $\mathcal{A}$ be a small abelian category. We have equivalence:*

$$\check{\mathbf{D}}(\mathrm{ind}\text{-}\mathcal{A}) \simeq \mathrm{ind}\text{-}\mathbf{D}^{\mathrm{b}}(\mathcal{A})$$

From now on, for a small abelian category $\mathcal{A}$, we will write $\bar{\mathcal{A}}$ for $\mathrm{ind}\text{-}\mathcal{A}$. Let $\check{\mathbf{D}}_{\geq 0}(\bar{\mathcal{A}})$ and $\check{\mathbf{D}}_{\leq 0}(\bar{\mathcal{A}})$ denote $\mathrm{ind}\text{-}\mathbf{D}^{\mathrm{b}}_{\geq 0}(\mathcal{A})$ and $\mathrm{ind}\text{-}\mathbf{D}^{\mathrm{b}}_{\leq 0}(\mathcal{A})$ respectively.

Theorem 10.1.3. *Let $\mathcal{A}$ be a small abelian category. The pair of subcategories $(\check{\mathbf{D}}_{\geq 0}(\bar{\mathcal{A}}), \check{\mathbf{D}}_{\leq 0}(\bar{\mathcal{A}}))$ form a t-structure of $\check{\mathbf{D}}(\bar{\mathcal{A}})$.*