Comma Categories and Filtered Categories

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This article is an effort to generalize the corresponding parts in Kashiwara-Schapira's book *Categories and Sheaves*, in which they consider the case of 1-categories. Besides that, the main reference is Lurie's *Higher Topos Theory*, abbreviated as HTT. We will always refer to the version on his personal website.

Contents

1 The Comma Construction of ∞-Categories

1.1 Comma Categories

Given a functor $f : \mathcal{C} \to \mathcal{D}$ between ∞-categories and an object $d \in \mathcal{D}$, we will write $\mathcal{C}_{/d}$ for $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d}$. We write $\mathcal{S}$ for the ∞-category of spaces.

Given any simplicial set K, we will write the composition $K \simeq K \times \{0\} \to K \times \Delta^1$ as i_0 and we can define i_1 in a similar way.

Definition 1.1.1. *Given a diagram of simplicial set:*

$$K' \xrightarrow{p} K \xleftarrow{q} K''$$

The comma object $\mathrm{M}(p, q)$ *is the simplicial set with* n*-simplexes as the following diagrams:*

$$\begin{array}{ccccc} \Delta^n & \xrightarrow{i_0} & \Delta^n \times \Delta^1 & \xleftarrow{i_1} & \Delta^n \\ \downarrow & & \downarrow & & \downarrow \\ K' & \xrightarrow{p} & K & \xleftarrow{q} & K'' \end{array}$$

Remark 1.1.2. *We have two natural maps and a simplicial homotopy:*

$$\begin{array}{ccccc} & & K' & & \\ & {\scriptstyle q'}\nearrow & & \searrow{\scriptstyle p} & \\ \mathrm{M}(p,q) & & \Downarrow & & K \\ & {\scriptstyle p'}\searrow & & \nearrow{\scriptstyle q} & \\ & & K'' & & \end{array}$$

The comma object is some kind of so-called lax 2-limit. Many important constructions in (∞-)category theory are special cases of comma object and let me exhibit an incomplete list:

$$\begin{aligned} \mathcal{C}^{c/} &\simeq \mathrm{M}(\Delta^0 \xrightarrow{c} \mathcal{C} \xleftarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C}) \\ \mathcal{C}^{/c} &\simeq \mathrm{M}(\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C} \xleftarrow{c} \Delta^0) \\ \mathcal{C}^{d/} &\simeq \mathrm{M}(\Delta^0 \xrightarrow{d} \mathcal{D} \xleftarrow{f} \mathcal{C}) \\ \mathcal{C}^{/d} &\simeq \mathrm{M}(\mathcal{C} \xrightarrow{f} \mathcal{D} \xleftarrow{d} \Delta^0) \\ \mathrm{map}_{\mathcal{C}}(x, y) &\simeq \mathrm{M}(\Delta^0 \xrightarrow{x} \mathcal{C} \xleftarrow{y} \Delta^0) \\ \mathcal{C}^{\Delta^1} &\simeq \mathrm{M}(\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C} \xleftarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C}) \\ \mathcal{C} \times \mathcal{D} &\simeq \mathrm{M}(\mathcal{C} \to \Delta^0 \leftarrow \mathcal{D}) \end{aligned}$$

They are actually isomorphic as simplicial sets.

Lemma 1.1.3. *If K is ∞-category, the functor $(p', q') : \mathrm{M}(p, q) \to K' \times K''$ is inner fibration.*

Proof. To fill in a diagram $\Lambda^n_i \to \mathrm{M}(p, q)$ $(1 \le i \le n)$:

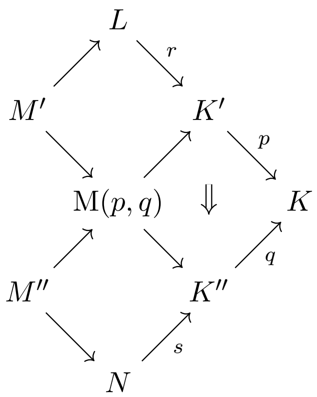
$$\begin{array}{ccccc} \Lambda^n_i & \xrightarrow{i_0} & \Lambda^n_i \times \Delta^1 & \xleftarrow{i_1} & \Lambda^n_i \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ K' & \xrightarrow{p} & K & \xleftarrow{q} & K'' \end{array}$$

We can fill in f' and f'' first (or use the given filling-in), then f. □

Corollary 1.1.4. *If K, K' and K'' are ∞-categories, the comma object $\mathrm{M}(p, q)$ is ∞-category and the functors p', q' are both inner fibrations.*

The next two lemmas are immediate consequences of the definition.

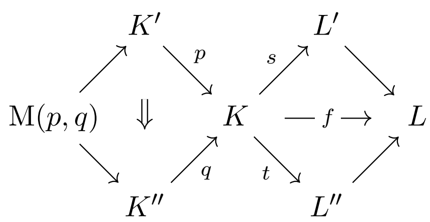
Lemma 1.1.5. *Given a comma object and two adjacent pullback squares:*



The canonical map is an isomorphism between simplicial sets:

$$f : M' \times_{\mathrm{M}(p,q)} M'' \to \mathrm{M}(pr, qs)$$

Lemma 1.1.6. *Given a comma object and a pullback square:*



We have a canonical pullback of simplicial sets:

$$\begin{array}{ccc} \mathrm{M}(p, q) & \longrightarrow & \mathrm{M}(sp, sq) \\ \downarrow & & \downarrow \\ \mathrm{M}(tp, tq) & \longrightarrow & \mathrm{M}(fp, fq) \end{array}$$

Corollary 1.1.7. *Given a homotopy pullback of ∞-categories:*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{a} & \mathcal{Y} \\ {\scriptstyle b}\downarrow & \searrow^{f} & \downarrow{\scriptstyle c} \\ \mathcal{Z} & \xrightarrow[d]{} & \mathcal{W} \end{array}$$

We have a homotopy pullback of mapping spaces:

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{X}}(x,y) & \longrightarrow & \mathrm{map}_{\mathcal{Y}}(a(x),a(y)) \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathcal{Z}}(b(x),b(y)) & \longrightarrow & \mathrm{map}_{\mathcal{W}}(f(x),f(y)) \end{array}$$

Proof. Assume that our homotopy pullack of ∞-categories is given by a pullback of simplicial sets and maps c, d are both categorical fibrations. Using the previous lemma and equivalence $\mathrm{map}_{\mathcal{X}}(x,y) \simeq \mathrm{M}(\Delta^0 \xrightarrow{x} \mathcal{X} \xleftarrow{y} \Delta^0)$, we have a pullback of mapping spaces (as simplicial sets) which is automatically homotopy pullback since the maps between mapping spaces are Kan fibrations (using the fact that c, d are categorical fibrations). $\square$

From now on, we will only take comma objects of ∞-categories, and it is safe since ∞-categories are closed under taking comma objects.

There are two alternative descriptions of $\mathrm{M}(\mathcal{C} \xrightarrow{p} \mathcal{D} \xleftarrow{q} \mathcal{E})$. One is the following pullback:

$$\begin{array}{ccc} \mathrm{M}(p,q) & \longrightarrow & \mathcal{D}^{\Delta^1} \\ \downarrow & & \downarrow \\ \mathcal{C} \times \mathcal{E} & \xrightarrow{(p,q)} & \mathcal{D} \times \mathcal{D} \end{array}$$

The other one is the limit of the following zig-zag:

$$\mathcal{C} \xrightarrow{p} \mathcal{D} \leftarrow \mathcal{D}^{\Delta^1} \rightarrow \mathcal{D} \xleftarrow{q} \mathcal{E}$$

All three descriptions give isomorphic simplicial sets. However, the previous two are homotopy limits as well. It is because the last two forms of limits are equivalent in any ∞-categories, and the functor $\mathcal{D}^{\Delta^1} \to \mathcal{D} \times \mathcal{D}$ is categorical fibration by HTT corollary 2.4.6.5. As a corollary, if the middle K is ∞-category, the comma object is invariant under categorical equivalence.

Remark 1.1.8. *We can deduce from the pullback description and corollary 1.1.7 that, the mapping space in comma category can be represented as homotopy pullback (between objects $A = (x, a, y)$ and $A' = (x', a', y')$):*

$$\begin{array}{ccc} \mathrm{map}_{\mathrm{M}(p,q)}(A,A') & \longrightarrow & \mathrm{map}_{\mathcal{C}}(x,x') \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathcal{E}}(y,y') & \longrightarrow & \mathrm{map}_{\mathcal{D}}(p(x),q(y')) \end{array}$$

Lemma 1.1.9. *Given a diagram of ∞-categories:*

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{p} & \mathcal{D} & \xleftarrow{q} & \mathcal{E} \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow \\ \mathcal{C}' & \xrightarrow[p']{} & \mathcal{D}' & \xleftarrow[q']{} & \mathcal{E}' \end{array}$$

It induces functor between comma categories:

$$F : \mathrm{M}(p,q) \to \mathrm{M}(p',q')$$

We have:

1. *If f', f and f'' are fully-faithful, F is fully-faithful.*
2. *If f', f'' are equivalences and f is fully-faithful, F is equivalence.*

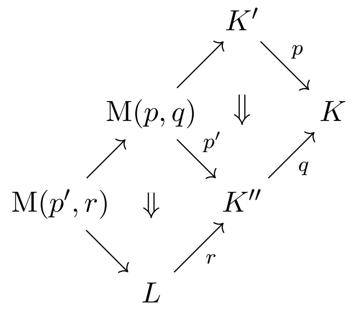
Proof. Given objects $A = (x,a,y)$ and $A = (x',a',y') \in \mathrm{M}(p,q)$, the mapping space between them can be represented as fiber product by the previous remark:

$$\mathrm{map}_{\mathrm{M}(p,q)}(A,A') \simeq \mathrm{map}_{\mathcal{C}}(x,x') \times_{\mathrm{map}_{\mathcal{D}}(p(x),q'(y'))} \mathrm{map}_{\mathcal{E}}(y,y')$$

Using this formula is enough to show (1). The claim (2) follows from the fact that, any morphism in $\mathcal{D}'$ between objects in the essential image of $\mathcal{C}'$ and $\mathcal{E}'$ is equivalent to a morphism in the essential image of $\mathcal{D}$. $\square$

1.2 Lax Fibers of Comma Categories

Theorem 1.2.1. *Given two adjacent comma categories of ∞-categories:*



The following canonical functor admits a right adjoint:

$$f : \mathrm{M}(p', r) \to \mathrm{M}(p, qr)$$

Proof. For the most convenience, we will write M and M$'$ for $\mathrm{M}(p', r)$ and $\mathrm{M}(p, qr)$ respectively. The object in M can be seen as tuple (x, a, y, b, z) such that $x \in K'$, $y \in K''$, $z \in L$ and $a : p(x) \to q(y)$, $b : y \to r(z)$. Similarly, the object in M$'$ can be seen as tuple (x', a', z') such that $x' \in K'$, $z' \in L$ and $a' : p(x') \to qr(z')$. The definition of f is roughly as taking (x, a, y, b, z) to $(x, q(b) \circ a, z)$. Write g for the proposed right adjoint of f. The definition of g is roughly as taking (x', a', z') to $(x', a', r(z'), \mathrm{id}_{r(z')}, z')$. There is a natural transformation $\mathrm{id} \to gf$ that its value at (x, a, y, b, z) is defined by the diagram:

$$\begin{array}{ccccccccc}
x & & p(x) & \xrightarrow{a} & q(y) & y & \xrightarrow{b} & r(z) & z \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle q(b)} & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \mathrm{id}} \\
x & & p(x) & \xrightarrow{q(b)\circ a} & qr(z) & r(z) & \xrightarrow{\mathrm{id}} & r(z) & z
\end{array}$$

We are left to check it is a unit transformation in the sense of HTT definition 5.2.2.7., that is, to show the following composition is an equivalence in the homotopy category of spaces (we write C for (x, a, y, b, z) and D for (x', a', z')):

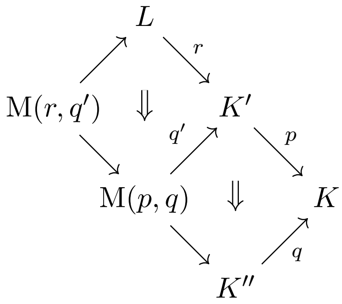
$$\mathrm{map}_{\mathrm{M}'}(f(C), D) \to \mathrm{map}_{\mathrm{M}}(gf(C), g(D)) \to \mathrm{map}_{\mathrm{M}}(C, g(D))$$

Basically, it is because both sides are equivalent to a fiber product:

$$\mathrm{map}_{K'}(x, x') \times_{\mathrm{map}_K(p(x), qr(z'))} \mathrm{map}_L(z, z')$$

$\square$

Remark 1.2.2. *If we combine the squares in the other direction:*

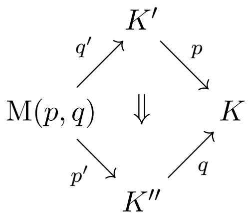


We still have a comparison functor:

$$f : \mathrm{M}(r, q') \to \mathrm{M}(pr, q)$$

This time, it admits a left adjoint.

Corollary 1.2.3. *Given a comma category of ∞-categories:*



If q is cofinal, q' is also cofinal.

Proof. Given any object $x \in K'$, the previous remark guarantees the natural functor $f : \mathrm{M}(p, q)_{x/} \to K''_{p(x)/}$ admits a left adjoint and therefore it is weak equivalence. We can use HTT theorem 4.1.3.1. to conclude the proof. □

Given a diagram of ∞-categories:

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{p} & \mathcal{D} & \xleftarrow{q} & \mathcal{E} \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow \\ \mathcal{C}' & \xrightarrow[p']{} & \mathcal{D}' & \xleftarrow[q']{} & \mathcal{E}' \end{array}$$

It induces functor between comma categories:

$$F : \mathrm{M}(p, q) \to \mathrm{M}(p', q')$$

Theorem 1.2.4. *Given an object $D = (x, a, y) \in \mathrm{M}(p', q')$, we have a categorical equivalence:*

$$\mathrm{M}(p, q)_{D/} \simeq \mathrm{M}(\mathcal{C}_{x/} \to \mathcal{D}_{p'(x)/} \leftarrow \mathcal{E}_{y/})$$

Proof. We have a pullback square:

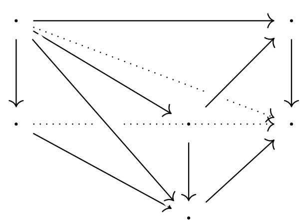
$$\begin{array}{ccc} (\mathcal{D}^{\Delta^1})_{a/} & \longrightarrow & (\mathcal{D}_{p'(x)/})^{\Delta^1} \\ \downarrow & & \downarrow \\ \mathcal{D}_{a/} & \longrightarrow & \mathcal{D}_{p'(x)/} \end{array}$$

It is because of a dual decomposition of simplicial sets:

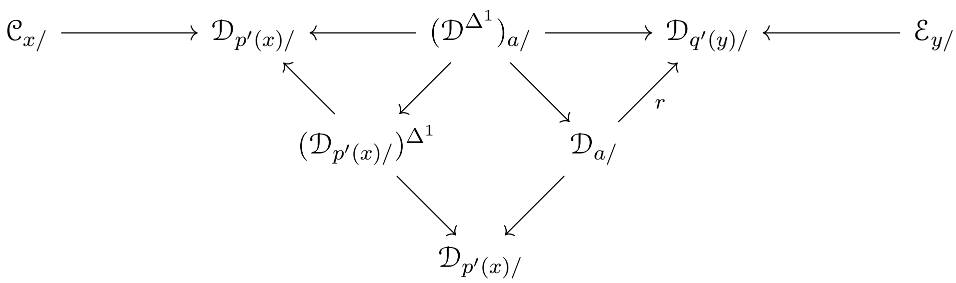
$$\Delta^1 \times \Delta^{n+1} \simeq \Delta^0 \star (\Delta^1 \times \Delta^n) \coprod_{\Delta^0 \star \Delta^n} \Delta^1 \star \Delta^n$$

The special case of $n = 1$ is drawn as follows:

$$\begin{array}{c} p'(x) \\ a \big\downarrow \\ q'(y) \end{array}$$



We have the following diagram:



Notice that the restriction r is trivial Kan fibration. The precise definition of the functor $\mathcal{E}_{y/} \to \mathcal{D}_{p'(x)/}$ appeared in this proposition should be the consecutive composition of $\mathcal{E}_{y/} \to \mathcal{D}_{q'(y)/}$, any section of r and $\mathcal{D}_{a/} \to \mathcal{D}_{p'(x)/}$. The limit of the first row is $\mathrm{M}(p, q)_{D/}$ by the very definition, and it is equivalent to the limit of the bottom border line because the middle square is pullback. And the last limit is categorical equivalent to $\mathrm{M}(\mathcal{C}_{x/} \to \mathcal{D}_{p'(x)/} \leftarrow \mathcal{E}_{y/})$ since r is trivial Kan fibration. □

Remark 1.2.5. *We have a dual version of the previous equivalence:*

$$\mathrm{M}(p, q)_{/D} \simeq \mathrm{M}(\mathcal{C}_{/x} \to \mathcal{D}_{/q'(y)} \leftarrow \mathcal{E}_{/y})$$

2 Filtered ∞-Categories

2.1 Cofinality and Filteredness

Lemma 2.1.1. *Given a filtered diagram of spaces $f : \mathcal{J} \to \mathcal{S}$, the following conditions are equivalent:*

1. *Its colimit $\varinjlim_{\alpha \in \mathcal{J}} f(\alpha)$ is contractible;*
2. *It satisfies the following two properties:*
   (a) *Given any $\alpha \in \mathcal{J}$, there exists morphism $\alpha \to \alpha'$ such that $f(\alpha')$ is non-empty;*
   (b) *Given any $\alpha \in \mathcal{J}$ and map $S^n \to f(\alpha)$, there exists morphism $\alpha \to \alpha'$ such that the composition $S^n \to f(\alpha) \to f(\alpha')$ is null-homotopic.*

Proof. We can consider the filtered systems of sets $\pi_0 f(-) : \mathcal{J} \to \mathbf{Set}$ and $\pi_0 \operatorname{map}_{\mathcal{S}}(S^n, f(-)) : \mathcal{J}_{\alpha/} \to \mathbf{Set}$. Since $*$ and S^n are compact objects of $\mathcal{S}$ and π_0 commutes with filtered limits, we can reduce the problem to properties of filtered colimits of sets.

1 $\Rightarrow$ 2: The claim follows from the fact that $\varinjlim_{\alpha \to \alpha' \in \mathcal{J}_{\alpha/}} \pi_0 \operatorname{map}_{\mathcal{S}}(S^n, f(\alpha')) \simeq *$ and $\varinjlim_{\alpha \in \mathcal{J}} \pi_0 f(\alpha) \simeq *$.

2 $\Rightarrow$ 1: The assumption implies that $\varinjlim_{\alpha \to \alpha' \in \mathcal{J}_{\alpha/}} \pi_0 \operatorname{map}_{\mathcal{S}}(S^n, f(\alpha')) \simeq *$ and $\varinjlim_{\alpha \in \mathcal{J}} \pi_0 f(\alpha) \simeq *$. Now use the criterion that a space $X \in \mathcal{S}$ is contractible if and only if $\pi_0 X$ is non-empty and $\pi_0 \operatorname{map}_{\mathcal{S}}(S^n, X) \simeq *$. $\square$

Lemma 2.1.2. *Given an ∞-category $\mathcal{C}$ such that for any $y \in \mathcal{C}$, $\mathcal{C}_{y/}$ is filtered, $\mathcal{C}$ itself is filtered if and only if for any objects $x, y \in \mathcal{J}$, the following filtered colimit is contractible:*

$$\varinjlim_{y \to y' \in \mathcal{C}_{y/}} \operatorname{map}_{\mathcal{J}}(x, y') \simeq *$$

Proof. The previous lemma shows that the assumptions in the criterion for filteredness given by HTT proposition 5.3.1.15. (together with HTT definition 5.3.1.1.) is equivalent our assumption. $\square$

We write $|K|$ for the geometric realization of simplicial set K.

Lemma 2.1.3. *Given a functor $f : \mathcal{J} \to \mathcal{C}$ between ∞-categories and object $x \in \mathcal{C}$, we have an equivalence:*

$$\varinjlim_{\alpha \in \mathcal{J}} \operatorname{map}_{\mathcal{C}}(x, f(\alpha)) \simeq |\, \mathcal{J}_{x/} \,|$$

Proof. The geometric realization of $\mathcal{J}_{x/}$ is equivalent to the homotopy colimit of the diagram $p : \mathcal{J} \to \mathcal{S}$ which is given by applying (reverse) Grothendieck construction to the left fibration $\mathcal{J}_{x/} \to \mathcal{J}$ (cf. HTT corollary 3.3.4.6.). The left fibration $\mathcal{J}_{x/} \to \mathcal{J}$ is pullback of $\mathcal{C}_{x/} \to \mathcal{C}$ along f, so by Yoneda lemma, we have $p(-) \simeq \mathrm{map}_{\mathcal{C}}(x, f(-))$. □

Theorem 2.1.4. *Given a functor $f : \mathcal{J} \to \mathcal{C}$ between ∞-categories such that $\mathcal{J}$ is filtered, the following conditions are equivalent:*

1. *f is cofinal;*
2. *For any $x \in \mathcal{C}$, $\mathcal{J}_{x/}$ is filtered;*
3. *The following properties hold:*
   - (a) *Given any $x \in \mathcal{C}$, there exists morphism $x \to f(j)$;*
   - (b) *Given any $x \in \mathcal{C}$, $j \in \mathcal{J}$ and map $S^n \to \mathrm{map}_{\mathcal{C}}(x, f(j))$, there exists morphism $j \to j'$ such that the composition $S^n \to \mathrm{map}_{\mathcal{C}}(x, f(j)) \to \mathrm{map}_{\mathcal{C}}(x, f(j'))$ is null-homotopic.*

Proof. $2 \Rightarrow 1$: That $\mathcal{J}_{x/}$ is filtered implies it is contractible (cf. HTT lemma 5.3.1.20.), and we can use the criterion HTT theorem 4.1.3.1.

$1 \Leftrightarrow 3$: We have $\varinjlim_{\mathcal{J}} \mathrm{map}_{\mathcal{C}}(x, f(-)) \simeq |\,\mathcal{J}_{x/}\,|$ by the previous lemma, and the claim follows from lemma 2.1.1.

$3 \Rightarrow 2$: The mapping space in $\mathcal{J}_{x/}$ between $A : x \to f(j)$ and $A' : x \to f(j')$ can be represented by homotopy pullback:

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{J}_{x/}}(A, A') & \longrightarrow & * \\ \downarrow & & \downarrow A' \\ \mathrm{map}_{\mathcal{J}}(j, j') & \longrightarrow & \mathrm{map}_{\mathcal{C}}(x, f(j')) \end{array}$$

We can combine corollary 1.1.7 and HTT proposition 2.4.4.3. (2) to prove this fact. Back to the track, our strategy is to apply lemma 2.1.2. Notice that we have restriction functor $(\mathcal{J}_{x/})_{A'/} \to \mathcal{J}_{j'/}$ that is a trivial Kan fibration. Therefore we have to show that (A'' is the composition $x \to f(j') \to f(j'')$):

$$\varinjlim_{j' \to j'' \in \mathcal{J}_{j'/}} \mathrm{map}_{\mathcal{J}_{x/}}(A, A'') \simeq *$$

Using the above pullback square and the fact that filtered colimits commute with pullbacks in $\mathcal{S}$, it is enough to show $\varinjlim_{j' \to j'' \in \mathcal{J}_{j'/}} \mathrm{map}_{\mathcal{J}}(j, j'') \simeq *$ and $\varinjlim_{j' \to j'' \in \mathcal{J}_{j'/}} \mathrm{map}_{\mathcal{C}}(x, f(j'')) \simeq *$. We can use the previous lemma to compute these filtered colimits, provided the facts that $\mathcal{J}_{j'/} \to \mathcal{J}$ is cofinal (since $(\mathcal{J}_{j'/})_{j''/} \simeq \mathcal{J}_{j''/}$ and hence filtered) and the composition of two cofinal functors $\mathcal{J}_{j'/} \to \mathcal{J} \to \mathcal{C}$ is cofinal (cf. HTT proposition 4.1.1.3. (2)). □

Remark 2.1.5. *By lemma 2.1.7 below, under the assumptions of the previous theorem, $\mathcal{C}$ has to be filtered.*

The following theorem concerns some natural cofinality that arises from filtered ∞-categories and cofinal functors between them.

Theorem 2.1.6. *Given a diagram of filtered ∞-categories with cofinal functors p and q:*

$$\mathcal{J} \xrightarrow{p} \mathcal{J}' \xrightarrow{q} \mathcal{J}''$$

The following propositions hold:

1. *For any $j' \in \mathcal{J}'$, $\mathcal{J}_{j'/} \to \mathcal{J}$ is cofinal;*
2. *For any $j \in \mathcal{J}$, $\mathcal{J}_{j/} \to \mathcal{J}_{p(j)/}$ is cofinal;*
3. *For any $j'' \in \mathcal{J}''$, $\mathcal{J}_{j''/} \to \mathcal{J}'_{j''/}$ is cofinal;*
4. *For any morhphism $f : j'' \to q(j') \in \mathcal{J}''$, the induced functor $\mathcal{J}_{j'/} \to \mathcal{J}_{j''/}$ is cofinal.*

Proof. (1) Given $j \in \mathcal{J}$, we have trivial Kan fibration $(\mathcal{J}_{j'/})_{j/} \to \mathcal{J}_{j/}$ and therefore $(\mathcal{J}_{j'/})_{j/}$ is contractible.

(2) This is a special case of (4).

(3) Given $A : j'' \to q(j') \in \mathcal{J}'_{j''/}$, we have trivial Kan fibration $(\mathcal{J}_{j''/})_{A/} \to \mathcal{J}_{j'/}$ and by the cofinality of p we conclude that $(\mathcal{J}_{j''/})_{A/}$ is contractible.

(4) The morphism f can be seen as object of $\mathcal{J}'_{j''/}$. By (3), the functor $\mathcal{J}_{j''/} \to \mathcal{J}'_{j''/}$ is cofinal and therefore $(\mathcal{J}_{j''/})_{f/}$ is filtered by the previous theorem. We have trivial Kan fibration $r : (\mathcal{J}_{j''/})_{f/} \to \mathcal{J}_{j'/}$ and by (1), cofinal functor $r' : (\mathcal{J}_{j''/})_{f/} \to \mathcal{J}_{j''/}$. The functor appeared in claim (4) is defined by taking any section of r and composing it with r', and the result is cofinal. $\square$

We conclude this section by two criteria for filteredness.

Lemma 2.1.7. *Given a functor $f : \mathcal{J} \to \mathcal{J}'$ between ∞-categories, if $\mathcal{J}$ is filtered and f is cofinal, $\mathcal{J}'$ is also filtered.*

Proof. There is a characterization of filtered ∞-categories that they are precisely those ∞-categories by which colimits (of spaces) are indexed could commutes with finite limits (of spaces) (cf. HTT proposition 5.3.3.3.). Since cofinal functor keeps colimits invariant, if $\mathcal{J}$ has this property, $\mathcal{J}'$ also has this property. $\square$

Lemma 2.1.8. *Given a functor $f : \mathcal{J} \to \mathcal{J}'$ between ∞-categories, if $\mathcal{J}'$ is filtered and f is right exact (cf. HTT definition 5.3.2.1.), $\mathcal{J}$ is also filtered.*

Proof. The pullback of identity $\mathrm{id}_{\mathcal{J}'} : \mathcal{J}' \to \mathcal{J}'$ along f is the identity $\mathrm{id}_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$. By the definition of right exact functors, since $\mathcal{J}'$ is filtered, $\mathcal{J}$ is also filtered. $\square$

2.2 Comma Category of Filtered Categories

Lemma 2.2.1. *Given a diagram of* ∞*-categories such that* $\mathcal{C}$ *is contractible and* q *is cofinal:*

$$\mathcal{C} \xrightarrow{p} \mathcal{D} \xleftarrow{q} \mathcal{E}$$

Then the comma category $\mathrm{M}(p, q)$ *is contractible.*

Proof. By corollary 1.2.3, there exists cofinal functor $\mathrm{M}(p, q) \to \mathcal{C}$, and cofinal functors are weak equivalences by HTT proposition 4.1.1.3. (3). □

Theorem 2.2.2. *Given a diagram of* ∞*-categories:*

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{p} & \mathcal{D} & \xleftarrow{q} & \mathcal{I} \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow \\ \mathcal{C}' & \xrightarrow[p']{} & \mathcal{D}' & \xleftarrow[q']{} & \mathcal{I}' \end{array}$$

If it satisfies the following conditions:

1. $\mathcal{I}$ *and* $\mathcal{I}'$ *are filtered;*
2. q*,* q'*,* f' *and* f'' *are cofinal.*

The induced functor between comma categories is cofinal:

$$F : \mathrm{M}(p, q) \to \mathrm{M}(p', q')$$

Notice that under these assumptions, $\mathcal{D}$ and $\mathcal{D}'$ are both filtered by lemma 2.1.7, and f is cofinal by HTT proposition 4.1.1.3. (2).

Proof. Our strategy is to use theorem 1.2.4 to represent $\mathrm{M}(p, q)_{D/}$ ($D = (x, a, y)$) as comma category $\mathrm{M}(\mathcal{C}_{x/} \to \mathcal{D}_{p'(x)/} \leftarrow \mathcal{I}_{y/})$ and then the previous lemma to show its contractibility. Since f' is cofinal, $\mathcal{C}_{x/}$ is contractible. We are left to show that $\mathcal{I}_{y/} \to \mathcal{D}_{p'(x)/}$ is cofinal. By definition, this functor factors as $\mathcal{I}_{y/} \to \mathcal{I}_{q'(y)/} \to \mathcal{D}_{q'(y)/} \to \mathcal{D}_{p'(x)/}$ and hence it is cofinal as being composition of cofinal functors by theorem 2.1.6 (2), (3) and (4). □

Theorem 2.2.3. *Given a comma category of* ∞*-categories such that* $\mathcal{I}'$*,* $\mathcal{I}''$ *are filtered and* q *is cofinal:*

$$\begin{array}{ccccc} & & \mathcal{I}' & & \\ & {\scriptstyle q'}\nearrow & & \searrow{\scriptstyle p} & \\ \mathrm{M}(p, q) & & \Downarrow & & \mathcal{I} \\ & {\scriptstyle p'}\searrow & & \nearrow{\scriptstyle q} & \\ & & \mathcal{I}'' & & \end{array}$$

The following propositions hold:

1. $\mathrm{M}(p,q)$ *is filtered;*

2. The functors q', p' *and* (q',p') *from* $\mathrm{M}(p,q)$ *to* $\mathcal{J}'$, $\mathcal{J}''$ *and* $\mathcal{J}'\times\mathcal{J}''$ *are cofinal.*

Proof. (1) By lemma 2.1.8, it is enough to show that q' is right exact, namely to prove that for any $j'\in\mathcal{J}'$, $\mathrm{M}(p,q)_{/j'}$ is filtered. Using the following diagram:

$$\begin{array}{ccccc} \mathcal{J}' & \xrightarrow{p} & \mathcal{J} & \xleftarrow{q} & \mathcal{J}'' \\ {\scriptstyle \mathrm{id}_{\mathcal{J}'}}\downarrow & & \downarrow & & \downarrow \\ \mathcal{J}' & \longrightarrow & \Delta^0 & \longleftarrow & \Delta^0 \end{array}$$

Remark 1.2.5 shows that $\mathrm{M}(p,q)_{/j'}\simeq\mathrm{M}(\mathcal{J}'_{/j'}\xrightarrow{\bar{p}}\mathcal{J}\xleftarrow{q}\mathcal{J}'')$ ($\bar{p}$ is the composition $\mathcal{J}'_{/j'}\to\mathcal{J}'\xrightarrow{p}\mathcal{J}$). Then we use the following diagram:

$$\begin{array}{ccccc} \Delta^0 & \xrightarrow{p(j')} & \mathcal{J} & \xleftarrow{q} & \mathcal{J}'' \\ {\scriptstyle \mathrm{id}_{j'}}\downarrow & & {\scriptstyle \mathrm{id}_{\mathcal{J}}}\downarrow & & {\scriptstyle \mathrm{id}_{\mathcal{J}''}}\downarrow \\ \mathcal{J}'_{/j'} & \xrightarrow[\bar{p}]{} & \mathcal{J} & \xleftarrow[q]{} & \mathcal{J}'' \end{array}$$

Notice that $\mathrm{id}_{j'}$ is terminal object of $\mathcal{J}'_{/j'}$, so the left vertical map is cofinal. We can apply the previous theorem and it follows that the induced functor $\mathcal{J}''_{p(j')/}\to\mathrm{M}(\bar{p},q)\simeq\mathrm{M}(p,q)_{/j'}$ is cofinal. Theorem 2.1.4 (2) implies that $\mathcal{J}''_{p(j')/}$ is filtered. Then lemma 2.1.7 shows $\mathrm{M}(p,q)_{/j'}$ is filtered.

(2) The following are equivalences:

$$\begin{aligned} \mathcal{J}' &\simeq \mathrm{M}(\mathcal{J}'\to\Delta^0\leftarrow\Delta^0) \\ \mathcal{J}'' &\simeq \mathrm{M}(\Delta^0\to\Delta^0\leftarrow\mathcal{J}'') \\ \mathcal{J}'\times\mathcal{J}'' &\simeq \mathrm{M}(\mathcal{J}'\to\Delta^0\leftarrow\mathcal{J}'') \end{aligned}$$

We can use remark 1.2.5 and these equivalences to represent $\mathrm{M}(p,q)_{x/}$ (for $x\in\mathcal{J}'$, $\mathcal{J}''$ or $\mathcal{J}'\times\mathcal{J}''$) and use lemma 2.2.1 to show its contractibility. □

2.3 Lax Limit of Filtered ∞-Categories

This section begins with an introduction to a special kind of diagrams.

Lemma 2.3.1. *For simplicial set K, the following properties are equivalent:*

1. *It is categorically equivalent to a minimal ∞-category, which has only finitely many non-degenerate simplexes.*

2. *It is categorically equivalent to a finite minimal 1-category that the length of composable non-identity morphisms has finite upper bound.*

Proof. (2) $\Rightarrow$ (1) Using the description of the non-degenerate simplexes in nerve of 1-category.

(1) $\Rightarrow$ (2) Assume that $K \simeq \mathcal{C}$ and $\mathcal{C}$ is a minimal ∞-category, we only need to show that $\mathcal{C}$ is actually an 1-category, and then we can use the description of the non-degenerate simplexes in nerve of 1-category again to conclude the proof.

Notice that, if $\mathcal{C}$ satisfies (1) then for any objects $x, y \in \mathcal{C}$, $\mathrm{map}_{\mathcal{C}}(x, y)$ also satisfies (1). To prove this claim, we use the model $\mathrm{map}^{\mathbf{R}}_{\mathcal{C}}(x, y)$ (cf. discussion before HTT proposition 1.2.2.3.). A simplex $\Delta^n \to \mathrm{map}^{\mathbf{R}}_{\mathcal{C}}(x, y)$ is a simplex $\Delta^{n+1} \to \mathcal{C}$ satisfying some properties and we can see that if the latter is degenerate, the former is also degenerate (except when $n = 0$ and $x = y$, there is another possibility that the 0-simplex represents id_x). This is enough to show our claim.

If $\mathcal{C}$ is a Kan complex, it has to be a finite set. To show this, given any object $x \in \mathcal{C}$, if $\pi_1(X, x)$ is nontrivial, we can take some $\gamma : \Delta^1 \to \mathcal{C}$ to represent a non-trivial loop. We have a categorical equivalence:

$$\mathrm{Spine_n} \simeq \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod_{\Delta^{\{2\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \to \Delta^n$$

Take the map $\mathrm{Spine_n} \to \mathcal{C}$ which maps each $\Delta^{\{i,i+1\}}$ to γ, and then exdend it to Δ^n. The resulting n-simplex cannot be degenerate and hence it contradicts our assumption. We apply this observation to $\Omega^n \mathcal{C}$ and we find out that all higher homotopy groups of $\mathcal{C}$ are trivial. Finally, for the original $\mathcal{C}$, the previous discussion applied to $\mathrm{map}_{\mathcal{C}}(x, y)$ show it is a finite set. $\square$

Definition 2.3.2. *A simplicial set K is called very small if it satisfies the above two (equivalent) properties.*

Let $p : \mathcal{X} \to K$ be a cocartesian fibration of ∞-categories, we will write Sect_p for the ∞-category $\mathrm{Map}_{/K}(id, p)$ of sections of p. Our main result in the section is the following:

Theorem 2.3.3. *Let $p : \mathcal{X} \to K$ be a cocartesian fibration such that K is very small and all fibers of p are filtered. Then for any object $i \in K$, the evaluation functor $\mathrm{Sect}_p \to p^{-1}(i)$ is cofinal. In particular, Sect_p is filtered and hence non-empty.*

Before the proof, we should establish some lemmas. Given a small simplicial set K, a cocartesian fibration $p : \mathcal{X} \to K^{\triangleright}$ over its right cone, let us call the cone point X and the base change of p to K will be named p^0. We have the following square and natural transformation $\eta : ti^* \to dj^*$:

$$\begin{array}{ccccc} & & \mathrm{Sect}_{p^0} & & \\ & \nearrow^{i^*} & & \searrow^{t} & \\ \mathrm{Sect}_p & & \eta \Downarrow & & \mathrm{Fun}(K, p^{-1}(X)) \\ & \searrow_{j^*} & & \nearrow_{d} & \\ & & p^{-1}(X) & & \end{array}$$

Lemma 2.3.4. *The previous square induces equivalence:*

$$\mathrm{Sect}_p \simeq \mathrm{M}(\mathrm{Sect}_{p^0} \to \mathrm{Fun}(K, p^{-1}(X)) \leftarrow p^{-1}(X))$$

The functor i^* is the restriction of sections to K, j^* the restriction to cone point X and d the diagonal functor which sends an object to the constant diagram. The functor t is defined by cocartesion lifting by solving the following extension problem (with the requirement that for each $i \in K$, the image of $i \times \Delta^1$ in $\mathcal{X}$ should be p-cocartesian) and then restricting the diagonal map to $K \times \{1\}$.

$$\begin{array}{ccc} K \times \{0\} & \xrightarrow{s} & \mathcal{X} \\ \big\downarrow & \nearrow & \big\downarrow p \\ K \times \Delta^1 & \xrightarrow[q]{} & K^{\triangleright} \end{array}$$

The natural transformation is defined similarly by the following extension problem:

$$\begin{array}{ccc} K \times \Lambda^2_0 & \xrightarrow{s} & \mathcal{X} \\ \big\downarrow & \nearrow & \big\downarrow p \\ K \times \Delta^2 & \xrightarrow[q]{} & K^{\triangleright} \end{array}$$

Theorem 2.3.5. *Let $p : \mathcal{X} \to K$ be a cocartesian fibration such that K is very small and all fibers of p are filtered. Then we have:*

1. *The ∞-category Sect_p is filtered and hence non-empty;*

2. *For any object $i \in K$, the evaluation functor $\mathrm{Sect}_p \to p^{-1}(i)$ is cofinal.*

Proof. We can assume that K is equivalent to the nerve $\mathrm{N}(\mathcal{C})$ of some minimal 1-category. By definition we can find a maximal object of $\mathcal{C}$, namely an object X such that admits no morphism towards other object. Let $\mathcal{C}^0$ be the full subcategory consists of objects other than X, and $\mathcal{C}^0_{/X}$ the over-category $\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/X}$. We have a natural simplicial homotopy:

$$\begin{array}{ccccc} & & \mathcal{C}^0 & & \\ & \overset{q}{\nearrow} & & \overset{i}{\searrow} & \\ \mathcal{C}^0_{/X} & & \Downarrow & & \mathcal{C} \\ & \searrow & & \underset{X}{\nearrow} & \\ & & \Delta^0 & & \end{array}$$

Let us denote the pullback of p along i and qi as p^0 and p^0_X. We have a natural equivalence induced by this simplicial homotopy:

$$\mathrm{Sect}_p \simeq \mathrm{M}(\mathrm{Sect}_{p^0} \to \mathrm{Sect}_{p^0_X} \to \mathrm{Fun}(\mathcal{C}^0_{/X}, p^{-1}(X)) \leftarrow p^{-1}(X))$$

Let us denote the right-hand-side as M. The reason for our claim is, we have pushout of simplicial set:

$$\begin{array}{ccc} \mathrm{N}(\mathcal{C}^0_{/X}) & \longrightarrow & \mathrm{N}(\mathcal{C}^0) \\ \downarrow & & \downarrow \\ \mathrm{N}(\mathcal{C}^0_{/X})^{\triangleright} & \longrightarrow & \mathrm{N}(\mathcal{C}) \end{array}$$

This is also a homotopy pushout in Joyal model structure. Therefore we have equivalence:

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathcal{D})} \mathrm{Fun}((\mathcal{C}^0_{/X})^{\triangleright}, \mathcal{D})$$

The comma object in question can also be represented as:

$$M \simeq \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathcal{D})} \mathrm{Fun}(\mathcal{C}^0_{/X} \diamond \Delta^0, \mathcal{D})$$

By HTT proposition 4.2.1.2., we justify our claim.

Now we can do induction on the cardinality of the isomorphic-classes of objects in $\mathcal{C}$. We have the following diagram:

$$\begin{array}{ccccc} \mathrm{ind}\text{-}\mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) & \longrightarrow & \mathrm{ind}\text{-}\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathcal{D}) & \longleftarrow & \mathrm{ind}\text{-}\mathcal{D} \\ \downarrow & & f\downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{C}^0, \mathrm{ind}\text{-}\mathcal{D}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}^0_{/X}, \mathrm{ind}\text{-}\mathcal{D}) & \longleftarrow & \mathrm{ind}\text{-}\mathcal{D} \end{array}$$

By our previous discussion, the inductive assumption, lemma 1.1.9 and theorem 3.0.3 , we only need to show the middle vertical functor f is fully-faithful. Using HTT proposition 5.3.5.11., it is enough to show the essential image of the following inclusion consists of compact objects:

$$\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^0_{/X}, \mathrm{ind}\text{-}\mathcal{D})$$

The point is, $\mathrm{N}(\mathcal{C}^0_{/X})$ is a finite simplicial set, and hence the mapping space between two functors in $\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathrm{ind}\text{-}\mathcal{D})$ is a canonical finite limit of mapping spaces between the values of each functor. Using the fact that filtered colimits commute with finite limits, we conclude our proof. $\square$

3 Applications to Ind-Objects

In the last section of this article, we apply the results established in previous sections to ind-objects.

Lemma 3.0.1. *Given a filtered diagram* $p : \mathcal{J} \to \mathcal{C}$ *which represents* $X \in \operatorname{ind-}\mathcal{C}$*, the canonical functor* $\tilde{p} : \mathcal{J} \to \mathcal{C}_{/X}$ *is cofinal.*

Proof. We will use lemma 2.1.3 to show cofinality. That means we have to prove that given any $A : Y \to X \in \mathcal{C}_{/X}$, we have:

$$\varinjlim_{\alpha \in \mathcal{J}} \operatorname{map}_{\mathcal{C}_{/X}}(A, p(\alpha) \to X) \simeq *$$

The mapping space in $\mathcal{C}_{/X}$ between $A : Y \to X$ and $A' : Y' \to X$ can be represented as homotopy pullback:

$$\begin{array}{ccc} \operatorname{map}_{\mathcal{C}_{/X}}(A, A') & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle A} \\ \operatorname{map}_{\mathcal{C}}(Y, Y') & \xrightarrow{A' \circ -} & \operatorname{map}_{\mathcal{C}}(Y, X) \end{array}$$

Since Y is compact in $\operatorname{ind-}\mathcal{C}$, we have $\varinjlim_{\alpha \in \mathcal{J}} \operatorname{map}_{\mathcal{C}}(Y, p(\alpha)) \simeq \operatorname{map}_{\mathcal{C}}(Y, X)$, and the bottom map becomes equivalence after taking colimit, so its fiber is contractible. □

Theorem 3.0.2. *Given a morphism* $f : X \to X' \in \operatorname{ind-}\mathcal{C}$ *and two filtered diagrams* $p : \mathcal{J} \to \mathcal{C}$ *and* $q : \mathcal{J}' \to \mathcal{C}$ *that represents* X *and* X' *respectively, there exists filtered* ∞*-category* $\mathcal{J}''$*, cofinal maps* p' *and* q' *and natural transformation* $pp' \to qq'$ *that represents* f*:*

$$\begin{array}{ccccc} & & \mathcal{J} & & \\ & {\scriptstyle p'}\nearrow & & \searrow{\scriptstyle p} & \\ \mathcal{J}'' & & \Downarrow & & \mathcal{C} \\ & {\scriptstyle q'}\searrow & & \nearrow{\scriptstyle q} & \\ & & \mathcal{J}' & & \end{array}$$

Proof. We have a diagram such that $\tilde{q}$ is cofinal by the previous lemma:

$$\mathcal{J} \xrightarrow{\tilde{p}} \mathcal{C}_{/X} \xrightarrow{f_!} \mathcal{C}_{/X'} \xleftarrow{\tilde{q}} \mathcal{J}'$$

Then $\mathrm{M}(f_! \tilde{p}, \tilde{q})$ is what we want by theorem 2.2.3. □

Theorem 3.0.3. *Given a diagram of* ∞*-categories* $K' \xrightarrow{p} K \xleftarrow{q} K''$*, the following canonical functor is an equivalence:*

$$\operatorname{ind-}\mathrm{M}(K' \to K \leftarrow K'') \longrightarrow \mathrm{M}(\operatorname{ind-}(K' \to K \leftarrow K''))$$

Proof. Without loss of generality, we assume K', K and K'' are ∞-categories. Let us focus on the restriction first:

$$\mathrm{M}(K' \to K \leftarrow K'') \longrightarrow \mathrm{M}(\mathrm{ind}\text{-}(K' \to K \leftarrow K''))$$

This functor is fully-faithful by definition, and we will prove that its image consists of compact objects. Given $(x, a, y) \in \mathrm{M}(K' \to K \leftarrow K'')$ and $(x', a', y') \in \mathrm{M}(\mathrm{ind}\text{-}(K' \to K \leftarrow K''))$, the mapping space between them can be represented as fiber product (cf. remark 1.1.8):

$$\mathrm{map}_{\mathrm{ind}\text{-}K'}(x, x') \times_{\mathrm{map}_{\mathrm{ind}\text{-}K}(p(x), q(y'))} \mathrm{map}_{\mathrm{ind}\text{-}K''}(y, y')$$

Since $\mathrm{ind}\text{-}K' \to \mathrm{ind}\text{-}K$ and $\mathrm{ind}\text{-}K'' \to \mathrm{ind}\text{-}K$ preserve filtered colimits, the functor $\mathrm{map}_{\mathrm{ind}\text{-}K'}(X, -) \times_{\mathrm{map}_{\mathrm{ind}\text{-}K}(p(X), q(-))} \mathrm{map}_{\mathrm{ind}\text{-}K''}(Y, -)$ preserves filtered colimits. We finished our proof of compactness.

By HTT proposition 5.3.5.11., we are left to prove the canonical functor in our proposition is essentially surjective. Given $(x', a, y') \in \mathrm{M}(\mathrm{ind}\text{-}(K' \to K \leftarrow K''))$ and represented x' and y' by $f : \mathcal{J} \to K'$ and $g : \mathcal{J}' \to K''$ respectively. We can apply theorem 3.0.2 to the map $a : p(x') \to q(y')$, and hence we can assume $\mathcal{J} \simeq \mathcal{J}'$ and the map is given by a natural transformation $pf \to qg$. Notice that the latter form can be seen as a diagram $\mathcal{J} \to \mathrm{M}(K' \to K \leftarrow K'')$, and essential surjectivity follows. □

We have the following generalization of HTT proposition 5.3.5.15.

Lemma 3.0.4. *For simplicial set K, the following properties are equivalent:*

1. *It is categorically equivalent to a minimal ∞-category, which has only finitely many non-degenerate simplexes.*
2. *It is categorically equivalent to a finite minimal 1-category that the length of composable non-identity morphisms has finite upper bound.*

Proof. (2) $\Rightarrow$ (1) Using the description of the non-degenerate simplexes in nerve of 1-category.

(1) $\Rightarrow$ (2) Assume that $K \simeq \mathcal{C}$ and $\mathcal{C}$ is a minimal ∞-category, we only need to show that $\mathcal{C}$ is actually an 1-category, and then we can use the description of the non-degenerate simplexes in nerve of 1-category again to conclude the proof.

Notice that, if $\mathcal{C}$ satisfies (1) then for any objects $x, y \in \mathcal{C}$, $\mathrm{map}_{\mathcal{C}}(x, y)$ also satisfies (1). To prove this claim, we use the model $\mathrm{map}^{\mathbf{R}}_{\mathcal{C}}(x, y)$ (cf. discussion before HTT proposition 1.2.2.3.). A simplex $\Delta^n \to \mathrm{map}^{\mathbf{R}}_{\mathcal{C}}(x, y)$ is a simplex $\Delta^{n+1} \to \mathcal{C}$ satisfying some properties and we can see that if the latter is degenerate, the former is also degenerate (except when $n = 0$ and $x = y$, there is another possibility that the 0-simplex represents id_x). This is enough to show our claim.

If $\mathcal{C}$ is a Kan complex, it has to be a finite set. To show this, given any object $x \in \mathcal{C}$, if $\pi_1(X, x)$ is nontrivial, we can take some $\gamma : \Delta^1 \to \mathcal{C}$ to represent a non-trivial loop. We have a categorical equivalence:

$$\mathrm{Spine_n} \simeq \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod_{\Delta^{\{2\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \to \Delta^n$$

Take the map $\mathrm{Spine_n} \to \mathcal{C}$ which maps each $\Delta^{\{i,i+1\}}$ to γ, and then exdend it to Δ^n. The resulting n-simplex cannot be degenerate and hence it contradicts our assumption. We apply this observation to $\Omega^n \mathcal{C}$ and we find out that all higher homotopy groups of $\mathcal{C}$ are trivial. Finally, for the original $\mathcal{C}$, the previous discussion applied to $\mathrm{map}_{\mathcal{C}}(x, y)$ show it is a finite set. □

Definition 3.0.5. *A simplicial set K is called very small if it satisfies the above two (equivalent) properties.*

Theorem 3.0.6. *Given an ∞-category $\mathcal{D}$ and very small simplicial set K, the comparison functor is equivalence:*

$$\mathrm{ind}\text{-}(\mathcal{D}^K) \to (\mathrm{ind}\text{-}\mathcal{D})^K$$

Proof. We can assume that K is the nerve $\mathrm{N}(\mathcal{C})$ of some minimal 1-category. By definition we can find a maximal object of $\mathcal{C}$, namely an object X such that admits no morphism towards other object. Let $\mathcal{C}^0$ be the full subcategory consists of objects other than X, and $\mathcal{C}^0_{/X}$ the over-category $\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/X}$. We have a natural simplicial homotopy:

$$\begin{array}{ccccc} & & \mathcal{C}^0 & & \\ & \overset{p}{\nearrow} & & \searrow & \\ \mathcal{C}^0_{/X} & & \Downarrow & & \mathcal{C} \\ & \searrow & & \nearrow_{X} & \\ & & \Delta^0 & & \end{array}$$

We have a natural equivalence induced by this simplicial homotopy:

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{M}(\mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^0_{/X}, \mathcal{D}) \leftarrow \mathrm{Fun}(\Delta^0, \mathcal{D}))$$

Let us denote the right-hand-side as M. The reason for our claim is, we have pushout of simplicial set:

$$\begin{array}{ccc} \mathrm{N}(\mathcal{C}^0_{/X}) & \longrightarrow & \mathrm{N}(\mathcal{C}^0) \\ \big\downarrow & & \big\downarrow \\ \mathrm{N}(\mathcal{C}^0_{/X})^{\triangleright} & \longrightarrow & \mathrm{N}(\mathcal{C}) \end{array}$$

This is also a homotopy pushout in Joyal model structure. Therefore we have equivalence:

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathcal{D})} \mathrm{Fun}(\mathcal{C}^{0,\triangleright}_{/X}, \mathcal{D})$$

The comma object in question can also be represented as:

$$M \simeq \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathcal{D})} \mathrm{Fun}(\mathcal{C}^0_{/X} \diamond \Delta^0, \mathcal{D})$$

By HTT proposition 4.2.1.2., we justify our claim.

Now we can do induction on the cardinality of the isomorphic-classes of objects in $\mathcal{C}$. We have the following diagram:

$$\begin{array}{ccccc} \mathrm{ind}\text{-}\mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) & \longrightarrow & \mathrm{ind}\text{-}\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathcal{D}) & \longleftarrow & \mathrm{ind}\text{-}\mathcal{D} \\ \downarrow & & f\downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{C}^0, \mathrm{ind}\text{-}\mathcal{D}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}^0_{/X}, \mathrm{ind}\text{-}\mathcal{D}) & \longleftarrow & \mathrm{ind}\text{-}\mathcal{D} \end{array}$$

By our previous discussion, the inductive assumption, lemma 1.1.9 and theorem 3.0.3 , we only need to show the middle vertical functor f is fully-faithful. Using HTT proposition 5.3.5.11., it is enough to show the essential image of the following inclusion consists of compact objects:

$$\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^0_{/X}, \mathrm{ind}\text{-}\mathcal{D})$$

The point is, $\mathrm{N}(\mathcal{C}^0_{/X})$ is a finite simplicial set, and hence the mapping space between two functors in $\mathrm{Fun}(\mathcal{C}^0_{/X}, \mathrm{ind}\text{-}\mathcal{D})$ is a canonical finite limit of mapping spaces between the values of each functor. Using the fact that filtered colimits commute with finite limits, we conclude our proof. □